(a) Spectral representation for an operator \( \hat{A} \)

For an arbitrary ket \( |14\rangle \in \mathcal{H} \) we have that

\[
\hat{A} |14\rangle = \sum_n \sum_{i=1}^{g_n} \lambda_n |\mu_i\rangle \langle \mu_i |14\rangle,
\]

where the \( \{ |\mu_i\rangle \} \) are the eigenvectors of \( \hat{A} \):

\[
\hat{A} |\mu_i\rangle = \lambda_n |\mu_i\rangle, \quad n = 1, 2, 3, \ldots
\]

\( i = 1, 2, \ldots, g_n \) (degeneracy of eigenvalue \( \lambda_n \))

We assumed the set \( \{ |\mu_i\rangle \} \) to form a complete basis.

Then:

\[
\hat{A} |14\rangle = \sum_n \sum_{i=1}^{g_n} \lambda_n |\mu_i\rangle \langle \mu_i |14\rangle
\]

\[
= \sum_n \lambda_n \sum_{i=1}^{g_n} |\mu_i\rangle \langle \mu_i |14\rangle
\]

Since this equation is valid \( \forall |14\rangle \in \mathcal{H} \), it becomes an equation for the operators on both sides of the equation

\[
\hat{A} = \sum_n \lambda_n \sum_{i=1}^{g_n} |\mu_i\rangle \langle \mu_i |
\]

\[
\hat{A} = \sum_n \lambda_n \hat{P}_n,
\]

where \( \hat{P}_n = \sum_{i=1}^{g_n} |\mu_i\rangle \langle \mu_i | \) is the projection onto the eigensubspace \( \mathcal{E}_n \) associated with the eigenvalue \( \lambda_n \).
Note the conceptual appeal of our result: it means that the operator is completely defined in terms of its spectrum of eigenvalues, and the projectors \( P_n \).

The projectors are of course constructed from the eigenvectors with the nice property that they are independent of the arbitrary phase with which the eigenvectors are obtained. (from \( |u_{n_i}\rangle \) and \( e^{i\theta} |u_{n_i}\rangle \) we obtain the same projectors \( |u_{n_i}\rangle \langle u_{n_i}| \).)

b) The eigenvalues of an operator are an intrinsic property of that operator.

Suppose that we diagonalize the operator \( \hat{A} \) using a certain basis \( \{|u_{n_i}\rangle\} \) to do the algebra. The secular equation reads

\[
\text{Det} (A - \lambda I) = 0
\]

where \( A \) is the matrix representing \( \hat{A} \) in the basis \( \{|u_{n_i}\rangle\} \), and where \( \lambda \) refers to any of the \( N \) eigenvalues (representation is \( N \)-dimensional).

Suppose that we did the algebra using another basis \( \{|u'_{n_i}\rangle\} \) (also of dimension \( N \)). The secular equation would read

\[
\text{Det} (A' - \lambda' I) = 0
\]

Now, we learned in class that both bases are related by a unitary transformation given by the operator \( \hat{S} \), defined by the equation

\[
\hat{S} = \sum |u'_{n_i}\rangle \langle u_{n_i}|
\]
\[ A'_{ij} = \langle \nu_i | \hat{A} | \nu_j \rangle = \langle \nu_i | \hat{S}^+ \hat{A} \hat{S} | \nu_j \rangle \]

We set
\[ A'_{ij} = \langle \nu_i | \hat{A}' | \nu_j \rangle \]

\[ \Rightarrow \hat{A}' = \hat{S}^+ \hat{A} \hat{S} \]

Then:
\[ \text{Det} (A' - \lambda'I) = \text{Det} (\hat{S}^+ \hat{A} \hat{S} - \lambda'I) = 0 \]
\[ \Rightarrow \text{Det} [\hat{S}^+ (A - \lambda'I) \hat{S}] = 0 \quad (\hat{S}^+ \hat{S} = I) \]
\[ (\text{Det} \hat{S}^+) \text{Det} (A - \lambda'I) (\text{Det} \hat{S}) = 0 \quad (\text{Det} AB = \text{Det} A \times \text{Det} B) \]
\[ \Rightarrow \text{Det} (A - \lambda'I) \frac{\text{Det} (\hat{S}^+ \hat{S})}{\text{Det} \hat{S}^+ \hat{S}} = 0 \]
\[ \frac{\text{Det} (A - \lambda'I)}{\text{Det} (\hat{S}^+ \hat{S})} = \frac{1}{\text{Det} \hat{S}^+ \hat{S}} \]

\[ \Rightarrow \text{Det} (A - \lambda'I) = 0 \]

We have proved that the set \( \{ \lambda' \} \) satisfies the same secular equation as the set \( \{ \lambda \} \).

Thus:\[ \{ \lambda' \} = \{ \lambda \} \]

Q.E.D.
1. Glauber's formula is proved on pages 174-175 of Cohen-Tannoudji. The purpose of this exercise was to bring this useful result to everybody's attention. It will be used in the future. Make sure you understand it!

Note: there are more general formulae for the product $e^A e^B$ for the case in which $A$ and $B$ do not commute with their commutator. (They are, of course, more complicated than Glauber's formula.)
An operator $\hat{\Sigma}_x$ is represented in some basis \{(1, 0), (1, 1)\} by the matrix

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In order to be able to work with the operator $e^{i\alpha \hat{\Sigma}_x}$, we must first find the eigenvalues and eigenvectors of $\hat{\Sigma}_x$. Actually, for the present problem we only need the eigenvalues explicitly. They are obtained from the secular equation

$$\det (\hat{\Sigma}_x - \lambda \mathbf{I}) = 0,$$

or

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1.$$

In the basis of its eigenvectors, we have that the operator $\hat{\Sigma}_x$ is represented by the matrix

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note: The two matrices given above are different, yet they represent the same entity ($\hat{\Sigma}_x$).

From C-T, p.170, Eq. (27) it then follows that in the basis of the eigenvectors of $\hat{\Sigma}_x$, the matrix which represents the operator $e^{i\alpha \hat{\Sigma}_x}$ is given by:

$$e^{i\alpha \hat{\Sigma}_x} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$
Note that this equation was written down for a very specific basis!

Then:

\[ e^{i\alpha \hat{O}_x} = \begin{pmatrix} \cos \alpha + i \sin \alpha & 0 \\ 0 & \cos \alpha - i \sin \alpha \end{pmatrix} \]

\[ = \cos \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Again, in the basis in which we are operating, we can identify the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) with the matrix \( \hat{O}_x \). Thus,

\[ e^{i\alpha \hat{O}_x} = \cos \alpha \hat{I} + i \sin \alpha \hat{O}_x \]

Having completed the proof (worked out consistently in the basis of the eigenvectors of \( \hat{O}_x \)), we can now view our result as an operator equation:

\[ e^{i\alpha \hat{O}_x} = \cos \alpha \hat{I} + i \sin \alpha \hat{O}_x \]
Let us denote by \{11', 12'\} the two-dimensional basis given by the eigenvectors of \(\hat{S}_y\). In this basis we have that

\[
\hat{S}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (see p.12)

Furthermore, in this same basis the operator \(e^{i\alpha \hat{S}_y}\) is represented by the matrix

\[
e^{i\alpha \hat{S}_y} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} = \cos \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Consider next the operator \(\hat{S}_\mu = \lambda \hat{S}_x + \mu \hat{S}_y\).

We know the matrix which represents \(\hat{S}_\mu\) in the basis \{11', 12'\} used in the previous problem. We have that

\[
\hat{S}_\mu = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

\[
\hat{S}_\mu = \begin{pmatrix} \lambda - i\mu & 1 \\ i & i\mu \end{pmatrix}
\]
In the original basis \{11\rangle, 12\rangle\} of problem 2, p. 203, we have that
\[
\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

Eigenvalues of \(\hat{\sigma}_y\):
\[
\text{det}\left( \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} \right) = 0
\]

\(\Rightarrow A^2 + 1 = 0 \Rightarrow \lambda = \pm 1\)
We need a matrix representation for the operator $e^{i\beta \hat{\sigma}_n}$. The best way to proceed is to find the basis that diagonalizes $\hat{\sigma}_n$. Let us call $\{1 I\}, \{1 II\}$ the two eigenvectors of $\hat{\sigma}_n$. In the present problem we do not need them explicitly.

Secular equation: $\text{Det} \begin{pmatrix} -\alpha & 1-i\mu \\ 1+i\mu & \alpha \end{pmatrix} = 0$.

(We have denoted by $\{\alpha\}$ the eigenvalues of $\hat{\sigma}_n$.)

Then:

$\alpha^2 - (\lambda^2 + \mu^2) = 0$.

$\therefore \alpha^2 = \lambda^2 + \mu^2 \Rightarrow \alpha_{1,2} = \pm (\lambda^2 + \mu^2)^{1/2}$

In the particular case that $\lambda^2 + \mu^2 = 1$, we have that $\alpha = \pm 1$.

Thus, in the $\{1 I\}, \{1 II\}$ basis:

$\hat{\sigma}_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

In the same basis:

$e^{i\beta \hat{\sigma}_n} = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} = \cos \beta \hat{I} + \sin \beta \hat{\sigma}_n$. 

From Glauber's formula we have that
\[ e^{2i \hat{G}_x} = e^{i \hat{G}_x} e^{i \hat{G}_x} \]

\[ e^{2i \hat{G}_x} = (e^{i \hat{G}_x})^2 \]

Using the results of the previous problem we have that in the basis of the eigenvectors of \( \hat{G}_x \):
\[ (e^{i \alpha \hat{G}_x})^2 = e^{2i \alpha \hat{G}_x} = \cos 2\alpha \mathbb{I} + i \sin 2\alpha \hat{G}_x \]

Let us prove this differently. From the previous problem we have that
\[ e^{2i \alpha \hat{G}_x} = (e^{i \alpha \hat{G}_x})^2 = (\mathbb{I} \cos \alpha + i \hat{G}_x \sin \alpha)^2 \]

\[ = \mathbb{I} \cos^2 \alpha + 2i \sin \alpha \cos \alpha \hat{G}_x - \sin^2 \alpha \hat{G}_x^2 \]

\[ = \mathbb{I} \cos^2 \alpha + i \sin 2\alpha \hat{G}_x - \sin^2 \alpha \hat{G}_x^2 \]

In the basis of the eigenvectors of \( \hat{G}_x \):
\[ e^{2i \alpha \hat{G}_x} = \cos^2 \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin 2\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \sin^2 \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} \cos^2 \alpha - \sin^2 \alpha & 0 \\ 0 & \cos^2 \alpha - \sin^2 \alpha \end{pmatrix} + i \begin{pmatrix} \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha \end{pmatrix} \]
\[ e^{i2\alpha \sigma_x} = \begin{pmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{pmatrix} + i \begin{pmatrix} \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha \end{pmatrix} \]

\[ e^{i2\alpha \sigma_x} = \cos 2\alpha I + i \sin 2\alpha \sigma_x \]

The operator \( e^{i(\hat{\sigma}_x + \hat{\sigma}_y)} \) is a particular case of the operator \( e^{i\hat{\sigma}_z} \) discussed earlier. It can be easily verified that

\[ e^{i(\hat{\sigma}_x + \hat{\sigma}_y)} \neq e^{i\hat{\sigma}_x} e^{i\hat{\sigma}_y} \]

\[ \text{Cote: It is easy to prove, using, for example, the orthonormal basis \{11, 12, 21, 22\}, that} \]

\[ \sigma_x \sigma_y - \sigma_y \sigma_x = 2i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv 2i \sigma_z \]

One can also verify that \( [\sigma_x, \sigma_z] = 0 \), and also \( [\sigma_y, \sigma_z] = 0 \). Thus, Clebsch-Gordan's formula cannot be used to relate the operators \( e^{i(\hat{\sigma}_x + \hat{\sigma}_y)} \) and \( e^{i\hat{\sigma}_x} e^{i\hat{\sigma}_y} \).
\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})
\]

\[
\hat{H}\lvert \psi_n \rangle = E_n \lvert \psi_n \rangle
\]

\[
[\hat{x}, \hat{p}] = i\hbar \hat{I}
\]

\[
[\hat{x}, \hat{H}] = \frac{1}{2m} [\hat{x}, \hat{p}^2] = \frac{1}{2m} \left( \hat{p} [\hat{x}, \hat{p}] + [\hat{x}, \hat{p}] \hat{p} \right) =\frac{i\hbar}{m}
\]

\[
[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}
\]

We then have that

\[
\langle \psi_n \lvert \hat{p} \lvert \psi_n \rangle = \frac{m}{i\hbar} \langle \psi_n \lvert \hat{x} \hat{H} - \hat{H} \hat{x} \lvert \psi_n \rangle
\]

\[
= \frac{m}{i\hbar} \left( E_n - E_n \right) \langle \psi_n \lvert \hat{x} \lvert \psi_n \rangle
\]

Remember this result: For the above Hamiltonian, the matrix elements of the momentum operator between energy eigenkets are proportional to those of the position operator.

\[
\langle \psi_n \lvert \hat{p}^2 \lvert \psi_n \rangle = \langle \psi_n \lvert \hat{p} \hat{p} \lvert \psi_n \rangle
\]

\[
\text{use closure relation}
\]

\[
\sum_{n=1}^{\infty} \langle \psi_n \lvert \hat{p} \hat{p} \lvert \psi_n \rangle = \frac{1}{\hbar^2}
\]
\[ <\phi_n|\hat{\rho}^2|\phi_n> = \]
\[ = \frac{1}{n'} \left( \frac{m}{i'\hbar} \right) (E_n - E_{n'}) <\phi_{n'}|\hat{\rho}^2|\phi_{n'}> \times \]
\[ \times \left( -\frac{m}{i'\hbar} \right) (E_n - E_{n'}) <\phi_{n'}|\hat{\rho}^2|\phi_{n'}> \]
\[ = <\phi_n|\hat{\rho}^2|\phi_n> = <\hat{\rho}^2> = \]
\[ = \frac{m^2}{\hbar^2} \frac{1}{n'} (E_n - E_{n'})^2 |<\phi_{n'}|\hat{\rho}^2|\phi_{n'}>|^2 \]
\[ \hat{\mathcal{A}} |q_n\rangle = E_n |q_n\rangle \]

\[ a) \quad \langle q_n | [\hat{A}, \hat{\mathcal{A}}] |q_n\rangle = \langle q_n | \hat{\mathcal{A}} - \hat{A} \hat{\mathcal{A}} |q_n\rangle = \langle q_n | \hat{\mathcal{A}} |q_n\rangle = \langle q_n | \hat{A} |q_n\rangle = (E_n - E_n) \langle q_n | \hat{A} |q_n\rangle = 0 \quad (\hat{A} = \hat{A}^+) \]

\[ b) \quad \hat{A} = \frac{1}{2m} \hat{\beta}^2 + V(\hat{x}) \]

(Note: we will study motion in space later on in the course.)

\[ c) \quad [\hat{\mathcal{A}}, \hat{x}] = \left[ \frac{1}{2m} \hat{\beta}^2, \hat{x} \right] = \]

\[ = \frac{1}{2m} \left( \hat{\beta} [\hat{\beta}, \hat{x}] + [\hat{\beta}, \hat{x}] \hat{\beta} \right) \]

\[ = \frac{1}{2m} (-i\hbar) 2\hat{x} \]

\[ [\hat{A}, \hat{x}] = -\frac{i\hbar}{m} \hat{\beta} \]

\[ iv) \quad [\hat{A}, \hat{\beta}] = [V(\hat{x}), \hat{\beta}] \]

From the basic commutators \[ [\hat{x}, \hat{p}] = i\hbar \hat{I} \] we (will prove that (CT p. 172))

\[ [V(\hat{x}), \hat{\beta}] = i\hbar V'(\hat{x}) \]
(Note: if \( V(x) = V_0 x^\lambda \), then \( V(x) = \alpha V_0 x^{\lambda-1} \).)

We then have that

\[
[\hat{\mathbf{H}}, \hat{\mathbf{p}}] = i\hbar \mathbf{V}'(x)
\]

\[
(iii) \quad [\hat{\mathbf{H}}, \hat{\mathbf{x}} \hat{\mathbf{p}}] = \hat{\mathbf{x}} [\hat{\mathbf{H}}, \hat{\mathbf{p}}] + [\hat{\mathbf{H}}, \hat{\mathbf{x}}] \hat{\mathbf{p}}
\]

\[
= i\hbar (\hat{\mathbf{x}} \mathbf{V}'(x) - \frac{1}{m} \hat{\mathbf{p}}^2)
\]

\[
c) \quad \langle \varphi_n | \hat{\mathbf{p}} | \varphi_m \rangle = -\frac{m}{i\hbar} \langle \varphi_n | [\hat{\mathbf{H}}, \hat{\mathbf{x}}] | \varphi_m \rangle
\]

\[
= 0 \quad \text{from a.)}
\]

\[
d) \quad \langle \varphi_n | [\hat{\mathbf{H}}, \hat{\mathbf{x}} \hat{\mathbf{p}}] | \varphi_m \rangle = 0 = \langle \varphi_n | \hat{\mathbf{x}} \mathbf{V}'(x) - \frac{1}{m} \hat{\mathbf{p}}^2 | \varphi_m \rangle
\]

\[
\Rightarrow \langle \varphi_n \left| \frac{\hat{\mathbf{p}}^2}{m} \right| \varphi_m \rangle = \langle \varphi_n \left| \hat{\mathbf{x}} \mathbf{V}'(x) \right| \varphi_m \rangle
\]

\[
\Rightarrow \langle \varphi_n \left| \frac{\hat{\mathbf{p}}^2}{2m} \right| \varphi_m \rangle = \frac{1}{2} \langle \varphi_n \left| \hat{\mathbf{x}} \mathbf{V}'(x) \right| \varphi_m \rangle
\]
\( y : V(x) = V_0 x^2 \)

\[ \Rightarrow \quad V'(x) = 2 V_0 x^{2-1} \]

\[ \therefore \quad \Delta V'(x) = 2 V_0 x^2 = \Delta V(x) \]

Thus:

\[ \langle \phi_n \mid \frac{\hbar^2}{2m} \mid \phi_n \rangle = \frac{1}{2} \langle \phi_n \mid V(x) \mid \phi_n \rangle \]

or

\[ \langle \text{kinetic energy} \rangle_n = \frac{1}{2} \langle \text{potential energy} \rangle_n \]

(Vincent Theorem)

\[ \lambda = 2 \quad (\text{harmonic oscillator}) \]

\[ \langle \text{kinetic energy} \rangle_n = \langle \text{potential energy} \rangle_n \]