Change of Basis

Consider the operator

\[ \hat{S} = \sum_i v_i \langle u_i | \langle u_i | \]

where the sets \{ | u_i \rangle \} and \{ | v_i \rangle \} are two different bases of \( \mathcal{B} \).

We have that

\[ \hat{S} | u_i \rangle = \sum_j v_j \langle u_j | u_i \rangle = \sum_j v_j \delta_{ij} \]

Thus:

\[ \hat{S} | u_i \rangle = | v_i \rangle \]

in other words, the operator \( \hat{S} \) transforms the kets of the \{ | u_i \rangle \} basis (or the "old" basis) into those of the \{ | v_i \rangle \} basis (or the "new" basis).

In addition, we have that

\[ \langle u_i | \hat{S} | u_j \rangle = S_{ij} = \langle u_i | v_j \rangle \]

Consider the expansion of a ket \( | \psi \rangle \) in either basis:

\[ | \psi \rangle = \sum_i c_i | u_i \rangle \]

and

\[ | \psi \rangle = \sum_i c'_i | v_i \rangle \]
How are the "components" \( c_i' \) in the new basis related to those in the old basis? We have that

\[
c_i' = \langle v_i' | 1 \rangle \\
= \langle u_i' | \hat{S}^+ | 1 \rangle \\
= \sum_j \langle u_i' | \hat{S}^+ | u_j \rangle \langle u_j | 1 \rangle 
\]

Thus:

\[
c_i' = \sum_j (S^+)_{ij} c_j 
\]

Similarly, the matrix elements of an operator \( \hat{A} \) in the \( \{ 1 \nu \} \) basis, \( A_{ij}' \), are related to the corresponding matrix elements in the \( \{ 1 \nu \} \) basis as follows:

\[
A_{ij}' = \langle v_i' | \hat{A} | v_j' \rangle \\
= \langle u_i' | \hat{S}^+ \hat{A} \hat{S} | u_j \rangle \\
= \sum_{k} \langle u_i' | \hat{S}^+ | u_k \rangle \langle u_k | \hat{A} | u_j \rangle \langle u_j | \hat{S} | u_i \rangle 
\]

Thus:

\[
A_{ij}' = \sum_{k} (S^+)_{ik} A_{k \ell} S_{\ell j}
\]
We can also view the above result as follows:

\[
A'_{ij} = \langle \tilde{v}_i | \hat{A} | v_j \rangle \\
= \langle u_\nu | \hat{S}^+ \hat{A} \hat{S} | u_\nu \rangle \\
= \langle u_\nu | \hat{A}' | u_\nu \rangle \\
\]

where we have made the definition

\[
\hat{A}' = \hat{S}^+ \hat{A} \hat{S}
\]

This view of the transformation of the matrix elements corresponds to thinking of \( \hat{A} \) as being subjected to the similarity transformation \( \hat{A} \rightarrow \hat{A}' \).

Now:

\[
\hat{S}^+ = \sum_i (| u_\nu \rangle \langle u_\nu |)^+ = \sum_i | u_\nu \rangle \langle u_\nu |
\]

Thus

\[
\hat{S} \hat{S}^+ = \sum_{ij} | u_\nu \rangle \langle u_\nu | (| u_\nu \rangle \langle u_\nu |) | v_j \rangle \langle v_j |
\]

\[
= \sum_{ij} | u_\nu \rangle \langle u_\nu | | v_j \rangle \langle v_j |
\]

\[
= \delta_{ij}
\]
Thus
\[ \hat{S} \hat{S}^+ = \sum_i \hat{v}_i \langle v_i | v_i \rangle = \mathbb{I} . \]

Similarly,
\[ \hat{S}^+ \hat{S} = \sum_{i,j} \hat{v}_i \langle v_i | v_j \rangle \langle v_j | v_i \rangle = \sum_i \hat{v}_i \langle v_i | v_i \rangle = \mathbb{I} . \]

The transformation operator \( \hat{S} \) is an example of a unitary operator, defined by the equation
\[ \hat{U} \hat{U}^+ = \mathbb{I} . \]

Unitary operators play an important role in Quantum Mechanics. As we shall see, symmetry transformation are represented by unitary operators.
Note:

\[ c'_i = \langle \nu'_i | \Psi \rangle \]  \hspace{1cm} (1)

\[ \hat{S} | \nu'_i \rangle = i | \nu'_i \rangle \implies \langle \nu'_i | = \langle \nu'_i | \hat{S}^+ \]

In (1) we think that we change the basis, while the state \( | \Psi \rangle \) remains the same.

Now:

\[ c'_i = \langle \nu'_i | \hat{S}^+ | \Psi \rangle = \langle \nu'_i | \Psi' \rangle \]  \hspace{1cm} (2a)

Where

\[ | \Psi' \rangle = \hat{S}^+ | \Psi \rangle \]  \hspace{1cm} (2b)

In (2) we think that we transform the state via the inverse transformation \( \hat{S}^+ (= \hat{S}^{-1}) \), and we leave the basis unchanged.

Similarly:

\[ A'_{ij} = \langle \nu'_i | \hat{A} | \nu'_j \rangle \]  \hspace{1cm} (3)

In (3) we transform the basis, not the operator:

\[ A'_{ij} = \langle \nu'_i | \hat{S}^+ \hat{A} \hat{S}^+ | \nu'_j \rangle = \langle \nu'_i | \hat{A}' | \nu'_j \rangle \]  \hspace{1cm} (4a)

Where
\[ \hat{A}' = \hat{S} + \hat{A} \hat{S} \] (46)

In (4) we transform the operators, not the basis.