Stationary States

The energy eigenkets, i.e., the eigenkets of $\hat{H}$, play a very important role in Quantum Mechanics. To begin with, they must be known, and explicitly introduced, in the study of time evolution. In effect, in order to make any progress, we must be able to expand the initial state ket $|\psi(t_0)\rangle$ as

$$|\psi(t_0)\rangle = \sum_{n} c_n \ |E_n\rangle,$$

where

$$c_n = \langle E_n | \psi(t_0) \rangle,$$

and the $\{|E_n\rangle\}$ are the energy eigenkets (for simplicity, the notation suggests non-degenerate energy levels — a very uncommon situation in practice), i.e.,

$$\hat{H} |E_n\rangle = E_n |E_n\rangle.$$

We then have, for a time-independent Hamiltonian,

$$|\psi(t)\rangle = \hat{U}(t, t_0) \ |\psi(t_0)\rangle = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \sum_{n} c_n \ |E_n\rangle = \sum_{n} c_n \ e^{-\frac{i}{\hbar} E_n(t-t_0)} \ |E_n\rangle,$$

and the time-evolution problem is solved! What remains,
typically, is to evaluate an amplitude of the form
\[ \langle \psi | \mathcal{U}(t) \rangle = \sum_n c_n e^{\frac{i}{\hbar} E_n (t-t_0)} \langle \psi | E_n \rangle, \]
where \( | \psi \rangle \) is an appropriate final state.

Suppose that the initial state is an energy eigenket, i.e.,
\[ | \psi(t_0) \rangle = | E_n \rangle. \]
Then, for \( t > t_0 \)
\[ | \psi(t) \rangle = e^{-\frac{i}{\hbar} \mathcal{H} (t-t_0)} | E_n \rangle \]
\[ = e^{\frac{i}{\hbar} E_n (t-t_0)} | E_n \rangle \]
\[ = \text{phase} \times | \psi(t_0) \rangle. \]

Thus, if at some initial time we place the system (by a measurement of the energy) in one of the energy eigenkets, and \( \mathcal{H} \) is time-independent, it will "stay there" for ever, i.e., the state of the system will remain the same from then on. Any future measurement of \( \mathcal{H} \) will always yield the value \( E_n \) with certainty. This is nothing but the statement of conservation of energy.

We refer to the energy eigenkets as stationary states of most appropriate denomination. Note that this definition applies both to bound states and to states in the continuum part of the energy spectrum.
More on stationary states

Consider an observable $\hat{B}$ which may or may not commute with $\hat{A}$. For an arbitrary state $\mid \psi(t) \rangle$ we have that (setting $t_0 = 0$)

\[
\langle \hat{B} \rangle (t) = \langle \psi(t) \mid \hat{B} \mid \psi(t) \rangle = \langle \psi(0) \mid e^{\frac{i}{\hbar} \hat{A} t} \hat{B} e^{-\frac{i}{\hbar} \hat{A} t} \mid \psi(0) \rangle.
\]

This means value in general depends on $t$. However, if the initial state is an energy eigenket, i.e., if $\mid \psi(0) \rangle = \mid E_n \rangle$, say, then

\[
\langle \hat{B} \rangle (t) = \langle \psi(0) \mid \hat{B} \mid \psi(0) \rangle = \langle \hat{B} \rangle (t=0).
\]

This is a property of the state ket, not of $\hat{B}$!

In addition,

\[
\langle \hat{B}^2 \rangle (t) = \langle \psi(0) \mid e^{\frac{i}{\hbar} \hat{A} t} \hat{B}^2 e^{-\frac{i}{\hbar} \hat{A} t} \mid \psi(0) \rangle = \langle \psi(0) \mid \hat{B}^2 \mid \psi(0) \rangle = \langle \hat{B}^2 \rangle (t=0),
\]

if $\mid \psi(0) \rangle = \mid E_n \rangle$. Thus:

\[
\langle (\Delta \hat{B})^2 \rangle (t) = \langle (\Delta \hat{B})^2 \rangle (t=0).
\]
Thus when a system is placed in a stationary state at some initial time, the mean value and dispersion of all observables do not change in time. This is true even if the observable in question is not a constant of motion! (See below.) This means that if we measure $\hat{B}$, we would obtain the same mean value and the same dispersion at any time. Of course, if $\hat{B}$ is incompatible with $\hat{A}$, the time evolution is abruptly changed by the measurement.

**Constant of Motion**

An observable $\hat{A}$ is said to be a constant of motion if

$$[\hat{A}, \hat{H}] = 0.$$  

We can always construct a basis of $\mathcal{E}$ made up of eigenkets common to such an observable and $\mathcal{E}$. Let us denote such a basis by \{ $|E_n, A_m\rangle$ \} (in general more indices will be needed, if $\hat{A}$ and $\hat{H}$ are not a CSCO).

If the initial state $14(0)$ is one of the kets of this basis, then

$$14(t) = e^{\frac{-i\hat{H}t}{\hbar}} |E_n, A_m\rangle$$

$$= e^{\frac{-iE_nt}{\hbar}} |E_n, A_m\rangle$$

$$= \text{phase} \times 14(0).$$
Thus the state the system is in remains the same for all $t$. Since this state is an eigenvector of $\hat{A}$ with eigenvalue $a_m$, a measurement of $\hat{A}$ will give this eigenvalue with certainty. This is a statement of conservation of the physical variable symbolized by $\hat{A}$. Clearly, to say that $\hat{A}$ is a constant of the motion is a very appropriate definition.

If $\hat{A}$ is a constant of the motion, then for an arbitrary ket $|\psi(t)\rangle$

\[
\langle \hat{A} \rangle (t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle \\
= \langle \psi(0) | e^{\frac{i}{\hbar} \hat{A} t} \hat{A} e^{-\frac{i}{\hbar} \hat{A} t} | \psi(0) \rangle \\
= \sum_{n} \sum_{m} \langle \psi(0) | e^{\frac{i}{\hbar} E_\psi t} E_\psi | n m \rangle \langle n m | \hat{A} | n' m' \rangle \langle n' m' | e^{-\frac{i}{\hbar} E_{n' m'} t} 1 \rangle \langle 1 | \psi(0) \rangle \\
= \sum_{n} \sum_{m} e^{\frac{i}{\hbar} E_\psi t} e^{-\frac{i}{\hbar} E_{n} t} \langle \psi(0) | n m \rangle \langle n m | \hat{A} | n' m' \rangle \langle n' m' | 1 \rangle \langle 1 | \psi(0) \rangle
\]

But:

\[
\langle n m | \hat{A} | n' m' \rangle = a_m \delta_{m m'} \delta_{n n'}
\]

Thus

\[
\langle \hat{A} \rangle (t) = \sum_{n m} \langle \psi(0) | n m \rangle a_m \langle n m | 1 \rangle \langle 1 | \psi(0) \rangle
\]
If \([\hat{A}, \hat{H}] = 0\), then

\[
\langle \hat{A} \rangle (t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle = \langle \psi(0) | e^{\frac{i}{\hbar} \hat{H} t} \hat{A} e^{-\frac{i}{\hbar} \hat{H} t} | \psi(0) \rangle = \langle \psi(0) | \hat{A} | \psi(0) \rangle \]

This result is a property of \(\hat{A}\), not the state ket.
\[
\sum \alpha_m \langle 4(0) \mid \hat{\rho}_m \mid 4(0) \rangle.
\]

Using the spectral decomposition of \( \hat{A} \) we have that
\[
\langle \hat{A} \rangle (t) = \langle 4(0) \mid \hat{A} \mid 4(0) \rangle = \langle \hat{A} \rangle (0).
\]

Thus the mean value of a constant of motion is independent of time whether or not the initial state is a stationary state.

This result can also be proved as follows:

\[
\frac{d}{dt} \langle \hat{A} \rangle (t) = (\frac{d}{dt} \langle 4(t) \mid ) \hat{A} \mid 4(t) \rangle
\]
\[
+ \langle 4(t) \mid \hat{A} (\frac{d}{dt} 4(t) \rangle
\]

where we have assumed that \( \hat{A} \) does not depend on \( t \).

Using the Schrödinger equation (whose content is the same as the use of the time-evolution operator in the previous demonstration) we have that
\[
\frac{d}{dt} \langle \hat{A} \rangle (t) = -\frac{i}{\hbar} \langle 4(t) \mid \hat{H} \hat{A} \mid 4(t) \rangle
\]
\[
+ \frac{i}{\hbar} \langle 4(t) \mid \hat{A} \hat{H} \mid 4(t) \rangle.
\]

Thus
\[
i\hbar \frac{d}{dt} \langle \hat{A} \rangle (t) = \langle [\hat{A}, \hat{H}] \rangle.
\]