Consider the Hamiltonian

$$\hat{H} = \frac{\hbar^2}{2m} \hat{\beta}^2 + \frac{1}{2} \mu w^2 \hat{x}^2,$$

where the operators $\hat{x}$ and $\hat{p}$ satisfy the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \hat{1}.$$

The above Hamiltonian is the quantum mechanical version of the energy of a particle moving in a potential of the form $V(x) = \frac{1}{2} \mu w^2 x^2$.

We will solve the eigenvalue problem for $\hat{H}$ using a procedure based on operator methods. This procedure makes essential use of the commutation relation satisfied by $\hat{x}$ and $\hat{p}$.

Note: as we will see below, the state space of the problem is of infinite dimension. Thus the standard procedure of solving a similar equation, used before for spin systems, is not applicable to the harmonic oscillator.

We introduce two auxiliary operators $a$ and $a^\dagger$ by the equations

$$a = \left(\frac{\mu w}{2\hbar}\right)^{1/2} \hat{x} + i \left(\frac{1}{2\mu w\hbar}\right)^{1/2} \hat{p},$$

and

$$a^\dagger = \left(\frac{\mu w}{2\hbar}\right)^{1/2} \hat{x} - i \left(\frac{1}{2\mu w\hbar}\right)^{1/2} \hat{p}.$$

Note that $a^\dagger$ is the hermitian adjoint of $a$. 
\[ x = \left( \frac{\hbar}{2m} \right)^{\frac{1}{2}} (a^+ + a) \]

\[ \hat{p} = i \left( \frac{\hbar \omega}{2} \right)^{\frac{1}{2}} (a^+ - a) \]

We have that

\[ [a, a^+] = \]

\[ = \left( \frac{m \hbar \omega}{2 \hbar} \right)^{\frac{1}{2}} \left( -i \right) \left( \frac{1}{2m \hbar \omega} \right)^{\frac{1}{2}} [\hat{x}, \hat{p}] + \]

\[ + \left( \frac{1}{2m \hbar \omega} \right)^{\frac{1}{2}} \left( \frac{m \hbar \omega}{2 \hbar} \right)^{\frac{1}{2}} [\hat{p}, \hat{x}] = \]

\[ = \left( \frac{m \hbar \omega}{4m \hbar \omega \hbar^2} \right)^{\frac{1}{2}} \left( -i \right) \left( 2 \hbar \omega \right)^{\frac{1}{2}} + \left( \frac{m \hbar \omega}{4m \hbar \omega \hbar^2} \right)^{\frac{1}{2}} \left( i \right) \left( -2 \hbar \omega \right)^{\frac{1}{2}} \]

\[ = \frac{1}{2 \hbar} \hbar + \frac{i}{2 \hbar} \hbar = 1 \]

\[ [a, a^+] = \hat{1} \]

In terms of the operators \( a \) and \( a^+ \), the Hamiltonian can be written as

\[ \hat{H} = \hbar \omega \left( a^+ a + \frac{1}{2} \hat{F} \right) \]
Now the operator \( \hat{a}^\dagger \hat{a} \) plays an important role in the theory of the harmonic oscillator. Thus we make the definition:

\[
\hat{N} = \hat{a}^\dagger \hat{a}
\]

Note that \( \hat{N}^\dagger = (\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger \hat{a} = \hat{N} \)

In terms of \( \hat{N} \) we then have that

\[
\hat{A} = \hbar \omega (\hat{N} + \frac{1}{2}). \quad (\hat{H} = \hbar \omega (\hat{N} + \frac{1}{2}))
\]

It is clear that \( [\hat{A}, \hat{N}] = 0 \). He can then obtain a basis of eigenvectors common to \( \hat{N} \) and \( \hat{H} \). Hence, because of the simple relation that exists between them, we have that the eigenvectors of \( \hat{N} \) are automatically eigenvectors of \( \hat{H} \). That is, the Hamiltonian eigenvectors can be labeled by one index, referring to the eigenvalues of \( \hat{N} \) or \( \hat{H} \) (plus another index to account for eventual degeneracies).

We will then concentrate in solving the eigenvalue problem for \( \hat{N} \):

\[
\hat{N}|\psi; i\rangle = \lambda_{N}|\psi; i\rangle,
\]

where \( \lambda_N \) are the eigenvalues of \( \hat{N} \), and \( i \) allows for possible degeneracies. We will show below that the spectrum of \( \hat{N} \) is non-degenerate, thus the index \( i \) will eventually be dropped.
\[ \hat{N} \equiv a + a^+ \]

\[ \hat{N} = \frac{1}{4} \omega (\hat{N} + \frac{1}{2}) \]

\[
\begin{align*}
[\hat{N}, a^+] &= [a^+ a, a^+] = a^+[a, a^+] = a^+ \\
[\hat{N}, a] &= [a^+ a, a] = [a^+, a] a = -a \\
[\hat{N}, a^+] &= a^+ \\
[\hat{N}, a] &= -a
\end{align*}
\]

\[ \hat{N} |\uparrow\rangle = |\uparrow\rangle \]

\[ \hat{N} |\downarrow\rangle = (\hat{N} a^+ - a^+ \hat{N} + a^+ \hat{N}) |\downarrow\rangle = (a^+ + a^+ \downarrow) |\downarrow\rangle = (1 + \frac{1}{2})(a^+ |\downarrow\rangle) \]

\[ \hat{N} |\uparrow\rangle = (\hat{N} a^+ + a^+ \hat{N} + a^+ \hat{N}) |\uparrow\rangle = (a^+ + a^+ \uparrow) |\uparrow\rangle = (1 + \frac{1}{2})(a^+ |\uparrow\rangle) \]  

\[ \hat{N} |\uparrow\rangle = (\uparrow + \downarrow)(a^+ |\downarrow\rangle) \]

\[ \hat{N} |\downarrow\rangle = (\downarrow - \downarrow)(a^+ |\downarrow\rangle) \]

\[ \hat{N} |\downarrow\rangle = (\downarrow - \downarrow)(a^+ |\downarrow\rangle) \]

Eigenvalue of \( \hat{N} \) with eigenvalue \( n+1 \),

Eigenvalue of \( \hat{N} \) with eigenvalue \( n-1 \),
\[ \langle 14 \rangle > 0 \quad \text{equality holds for null vector} \]

\[
\langle 1 \rangle a + a 10 \rangle \geq 0
\]

\[
\Rightarrow \quad \frac{\langle 1 \rangle a + a 10 \rangle}{\geq 0}
\]

\[ \forall n \in \text{non-negative integer} \]

\[
\text{Suppose } n \leq \nu \leq n + 1
\]

\[
a^{\nu} 10 \rangle = \# x 1_{\nu-n} > \leq 1
\]

\[
a^{n+1} 10 \rangle = \text{number } x a 1 0_{\nu-n} >
\]

\[
= \text{number } x 1_{\nu-(n+1)} > \leq 0
\]

Above this would violate the result that the eigenvalues of \( N \) (which are the basis of an basis) are \( \geq 0 \).

Only way out of contradiction: \( \nu \) is an integer

In that case:

\[
a^n 10 \rangle = \text{number } 1 0 >
\]

From above:

\[
\| a 1 \rangle > \| = 0 \Rightarrow a 1 \rangle > = 0
\]
\[ \hat{H} |n\rangle = \varepsilon_n |n\rangle \quad \text{for} \quad n = 0, 1, 2, \ldots \]

\[ \varepsilon_n = \hbar \omega (n + \frac{1}{2}) \]

\( \langle \text{energy are quantized!} \rangle \)

\[ \langle \text{creates quantum} \hbar \omega \rangle \]

\[ \langle \text{destroys} \rangle \]

\[ \langle \text{creates} \rangle \]

\[ \langle \text{destroys} \rangle \]
We will prove separately (later on) that the ground state is un-degenerate.

We prove first that if $E_n$ is un-degenerate, then so is $E_{n+1}$.

Suppose, then:

$$\hat{N}|m\rangle = m|m\rangle \quad \text{(no additional level)}$$

$|m+1, i\rangle$ ? is the extra level needed?

$$a|m+1, i\rangle = \# |m\rangle = c_i |m\rangle$$

$$a^+ a|m+1, i\rangle = c_i a^+ |m\rangle$$

$$\Rightarrow (n+1) |m+1, i\rangle = c_i a^+ |m\rangle$$

Thus:

$$|m+1, i\rangle = \frac{c_i}{(n+1)} (a^+ |m\rangle)$$

\[\hat{\phi_i} = \frac{c_i}{(n+1)} \text{common.}\]

... all the $|m+1, i\rangle$ are proportional to each other \(\Rightarrow\) physically the same.
We will obtain them in terms of the ground state $|0\rangle$.

First:

\[
|a^+ n\rangle = c |n+1\rangle
\]

\[
\langle n | a^+ a | n \rangle = 1 c^2 \langle n+1 | n+1 \rangle
\]

\[
= a a^+ - a^+ a + a^2
\]

\[
= 1
\]

\[
\langle n | (i + \delta) | n \rangle = (n+1) \langle n | n \rangle = 1 c^2
\]

\[
c = \sqrt{n+1} \quad \text{with choice of phase}
\]

\[
|a^+ n\rangle = \sqrt{n+1} |n+1\rangle
\]

Next:

\[
|a n\rangle = c |n-1\rangle
\]

\[
\langle n | a a^+ | n \rangle = 1 c^2 \langle n-1 | n-1 \rangle
\]

\[
\approx 1
\]

\[
\approx 1
\]

\[
\Rightarrow c = \sqrt{n} \quad \text{with choice of phase}
\]
Then:

\[ |m+1\rangle = \frac{1}{\sqrt{m+1}} a^+ |m\rangle \]

\[ |1\rangle = a^+ |0\rangle \]

\[ |2\rangle = \frac{1}{\sqrt{12}} a^+ |1\rangle = \frac{1}{\sqrt{12}} (a^+)^2 |0\rangle \]

\[ |3\rangle = \frac{1}{\sqrt{13}} a^+ |2\rangle = \frac{1}{\sqrt{13}} (a^+)^3 |0\rangle \]

\[ \text{Ansatz} \]

\[ |m-1\rangle = \frac{1}{\sqrt{(m-1)!}} (a^+)^{m-1} |0\rangle \]

Then:

\[ |m\rangle = \frac{1}{\sqrt{m}} a^+ |m-1\rangle = \frac{1}{\sqrt{m}} \frac{1}{\sqrt{(m-1)!}} (a^+)^{m-1} |0\rangle \]

\[ \therefore |m\rangle = \frac{1}{\sqrt{m!}} (a^+)^m |0\rangle \]
Hermitian, and its eigenvalues are real. Beginning

\[ \langle n | m' \rangle = \delta_{nm} \] 

normalization.

\[ \sum_{n=0}^{\infty} \langle n | n \rangle = 1 \] 

closure.

Notice that dimension of space \( |N \rangle \) is infinite.