It takes work to lift a mass in a gravitational field. This work (units: force \cdot distance = N \cdot m) increases the gravitational potential energy.
It takes work to lift a charge in an electric field.

This increases the charge's electric potential energy.

\[ \Delta W = \bar{F} \cdot \Delta \bar{x} \]

Unit are Newtons \(\cdot\) Meters = Joules

\[ W_{a \rightarrow b} = \int_{a}^{b} \bar{F} \cdot d\bar{r} = \int_{a}^{b} F \cos \theta dl \]

Forces like Gravity and the Electric Force are **Conservative Forces** [i.e. No Friction] Work done by conservative forces can be described as a change in **Potential Energy**

\[ W_{a \rightarrow b} = (U_a - U_b) = -(U_b - U_a) = -\Delta U \]

\(\Delta U\) can be positive or negative

\(\Delta U\) positive - Work is “done” in moving from point a to point b
\(\Delta U\) negative - Energy is “released” in moving from point a to point b

To avoid confusion it is often helpful to think about a ball rolling up or down a hill -

\(\Delta U\) positive - Work is “done” in rolling the ball UP the hill - higher up means higher \(E_p\)
\(\Delta U\) negative - Energy is “released” when the ball rolls down - lower down means lower \(E_p\)
Electrostatic Potential Energy in a Uniform Electric Field

Move a positive charge in a vertical direction \( d\vec{l} = \hat{y}dl \)

\[
W = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} -qE\hat{y} \cdot \hat{y}dl = \int_{a}^{b} -qEdl = -qE(y_{b} - y_{a}) = -qE\Delta y
\]

\[W = -\Delta U \quad \Rightarrow \quad \Delta U_{a\rightarrow b} = qE\Delta y\]

Looks like \( U_{\text{gravity}} = mg\Delta h \)

Now move from \( b \rightarrow c \)

But \( \vec{F} \) is perpendicular to \( d\vec{l} \)

So no work is done and there is no change in \( U \)

\[\Delta U_{a\rightarrow c} = qE\Delta y\]
Now, instead, go directly from $a \rightarrow c$ and compute $\Delta U$

\[ W = \int_{a}^{c} \vec{F} \cdot d\vec{l} = \int_{a}^{c} F \cos(\phi) dl \]

\[ W = -qE \cos(\phi) \int_{a}^{c} dl \]

\[ W = -qE \cos(\phi)L \]

Path length is given by:

\[ y = L \cos(\phi) \rightarrow L = \frac{\Delta y}{\cos(\phi)} \]

Therefore:

\[ W = -qE \Delta y \]

Work done on a charge in an electric Field is INDEPENDENT OF PATH.

Since $W_{a \rightarrow b} = (U_a - U_b) = -(U_b - U_a) = -\Delta U$ is independent of path, all that matters are the values of the potential energy at the start and At the finish.
Work and Potential Energy - Keeping the signs straight

\[ W_{a \to b} = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} F \cos \phi \, dl \]

Work done by a Force

\[ W_{a \to b} = U_{a} - U_{b} = -(U_{b} - U_{a}) = -\Delta U \]
(work done by a conservative force)

(a) Positive charge moves in direction of \( \vec{E} \): field does positive work on charge, potential energy \( U \) decreases

(b) Positive charge moves in direction opposite \( \vec{E} \): field does negative work on charge, potential energy \( U \) increases

(a) Negative charge moves in direction of \( \vec{E} \): field does negative work on charge, potential energy \( U \) increases

(b) Negative charge moves in direction opposite \( \vec{E} \): field does positive work on charge, potential energy \( U \) decreases
Work and Potential Energy

\[ W_{a\rightarrow b} = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} F \cos \phi \, dl \]

Work done by a Force

\[ W_{a\rightarrow b} = U_{a} - U_{b} = -(U_{b} - U_{a}) = -\Delta U \]

(work done by a conservative force)

The sign is critically important

The Work in the equations above refers to the Work DONE BY THE ELECTRIC FIELD

If I slowly move the charge, * I will always be applying a force which is equal and opposite to the Electrostatic Force. The work DONE BY ME will be equal and opposite to the work done by the E field.

* don’t change in kinetic energy
The Path of Integration Does NOT matter for a "conservative" force such as the Electrostatic Force

\[ W_{a \rightarrow b} = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} F \cos \phi \, dl \]
Electrostatic Potential Energy of Two Point Charges

Test charge $q_0$ moves from $a$ to $b$ along radial line from $q$

\[ F_r = \frac{qq_0}{4\pi \varepsilon_0 r^2} \]

\[
W_{a \rightarrow b} = \int_a^b F_r \, dr = \int_a^b \frac{qq_0}{4\pi \varepsilon_0 r^2} \, dr = \frac{qq_0}{4\pi \varepsilon_0} \int_a^b \frac{1}{r^2} \, dr = \frac{qq_0}{4\pi \varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) 
\]

\[
W_{a \rightarrow b} = U_a - U_b = -\Delta U = \frac{qq_0}{4\pi \varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) 
\]
The Path of Integration Does NOT matter for a "conservative" force such as the Electrostatic Force

\[ W_{a \rightarrow b} = \int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b F \cos \phi \, dl \]

\[ W_{a \rightarrow b} = U_a - U_b = -\Delta U = \frac{qq_0}{4\pi \varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \]
There is NO ABSOLUTE ZERO for Potential Energy
It is **CHANGE** in Potential Energy That is Important

We can **Pick any Distance to be the “Baseline” for Potential Energy**

For "Convenience" we pick the zero of potential energy to refer the Energy of the charges when they are VERY far away from each other
We pick ZERO for potential energy to be "Infinite Separation"

\[-\Delta U = \frac{qq_0}{4\pi\varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right)\]

Let \( r_a = \infty \)

\[-\Delta U = \frac{qq_0}{4\pi\varepsilon_0} \left( 0 - \frac{1}{r_b} \right)\]

Define \( U(r) = \frac{qq_0}{4\pi\varepsilon_0} \frac{1}{r} \)

(a) \( q, q_0 \) have same sign

(b) \( q, q_0 \) have opposite signs
The Potential Energy:

\[ U(r) = \frac{qq_0}{4\pi\varepsilon_0} \frac{1}{r} \]

is related to the amount of work that must be done to bring two charges together from far away.

I must do POSITIVE work to bring two like charges together

I must do NEGATIVE work to bring two like charges together
Electrostatic Potential Energy for Several Points Charges

What is the electrostatic potential energy of $q_0$ due to $q_1$, $q_2$, and $q_3$?

**Electrostatic Potential Energy is a SCALAR and adds algebraically**

This is different from the Electric Field; a VECTOR that adds vectorially

$$U = \frac{q_0}{4\pi\varepsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots \right) = \frac{q_0}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}$$
(point charge $q_0$ and collection of charges $q_i$)
Sample Problem #1

\[ q_1 = -e \quad q_2 = +e \quad q_3 = +e \]

\[ \begin{array}{c}
 q_1 \\
 x = 0 \\
 q_2 \\
 x = a \\
 q_3 \\
 x = 2a \\
\end{array} \]

What is the potential energy of charge \( q_1 \) due in the field of \( q_2 \) AND \( q_3 \)?

\[
U = \frac{q_1}{4\pi\varepsilon_0} \left( \frac{q_2}{a} + \frac{q_3}{2a} \right) = \frac{-e}{4\pi\varepsilon_0} \left( \frac{e}{a} + \frac{e}{2a} \right)
\]

\[
= \frac{-e^2}{4\pi\varepsilon_0} \left( \frac{3}{2a} \right)
\]

Check sign - Sign is negative which is correct for opposite charges. System is at a lower potential energy than if \( a \) was "at \( \infty \)"
Sample Problem #2

What is the potential energy of charge $q_2$ in the field of $q_1$ and $q_3$?

$$U = \frac{q_2}{4\pi \varepsilon_0} \left( \frac{q_1}{a} + \frac{q_3}{a} \right) = \frac{e}{4\pi \varepsilon_0} \left( \frac{-e}{a} + \frac{e}{a} \right) = 0$$

This is consistent with the amount of work required to "bring" $q_2$ in from $\infty$. The $y$ components cancel.
The Electrostatic Potential Energy of a point charge $q_0$ and a collection of Charges $q_i$:

$$U = \frac{q_0}{4\pi\epsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots \right) = \frac{q_0}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i}$$

(point charge $q_0$ and collection of charges $q_i$)
Sample Problem #3

What is the total potential energy of this charge distribution?

Start by removing ALL charges to $\infty$. This corresponds to the "zero" of Potential Energy.
Step 1: All Charges at $\infty$

$U = 0$

Step 2: Bring Charge $q_3$ in from infinity

$U = 0$

This requires no work because there are no other charges present.
Step 3: Bring Charge $q_2$ in from infinity,

\[ q_2 = +e \quad q_3 = +e \]

\[ U = \frac{1}{4\pi\varepsilon_0} \left( \frac{q_2 q_3}{r_{23}} \right) \]

Step 4: Bring Charge $q_1$ in from infinity,

\[ q_1 = -e \quad q_2 = +e \quad q_3 = +e \]

\[ U = \frac{1}{4\pi\varepsilon_0} \left( \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} \right) \]
The Electrostatic Potential Energy of a Collection of Point Charges:

\[
U = \frac{1}{4 \pi \varepsilon_0} \sum_{i<j} \frac{q_i q_j}{r_{ij}}
\]

Note: BE CAREFUL NOT TO DOUBLE COUNT
We defined the concept of the Electric Field as a generalization of the Electric Field.

The Electric Field assigns a VECTOR to each point in Space.
Electrostatic Potential Energy of a charge $q_0$ and a collection of charges $q_i$

\[
U = \frac{q_0}{4\pi\varepsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots \right) = \frac{q_0}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}
\]

If the charges $q_i$ remain fixed and I move $q_0$, I can express Potential Energy as a Function of the position of $q_0$

\[
U(r) = \frac{q_0}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}
\]
We defined the concept of the Electric Potential as a generalization of the Electric Potential Energy.

The Electric Field assigns a Scalar to each point in Space
Electrostatic Potential

\[ U(r) = \frac{q_0}{4\pi\varepsilon_0} \frac{q}{r} \quad V(r) = \frac{U(r)}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \]

Electrostatic Potential of POINT CHARGE

Electrostatic Potential of set of POINT CHARGES

IMPORTANT: Like Work - Electrostatic Potential is a “relative” quantity.

\[ W_{a \rightarrow b} = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} q_0 \vec{E} \cdot d\vec{l} \]

\[ V_a - V_b = \int_{a}^{b} \vec{E} \cdot d\vec{l} = \int_{a}^{b} E \cos \phi dl \]

UNITs – Volt = Joule/Coulomb
$\Delta V = 1.5 \text{ volts}$
Example

Find the potential at any height $y$ between two oppositely charged parallel plates

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E \cos \phi \, dl$$

$$V(y) = \frac{U(y)}{q_0} = \frac{q_0 E_y}{q_0} = E_y$$

We have chosen $V=0$ at point $b$
Example

**Conducting Sphere Revisited**

![Diagram showing electric field and potential for a conducting sphere]

**Important Note** – A conducting surface is always an equipotential surface
Equipotential Surfaces and Electric Field Lines

Equipotential Surfaces are always perpendicular to Electric Field Lines.

(a) A single positive charge

(b) An electric dipole

(c) Two equal positive charges

Equipotential Surfaces are always perpendicular to Electric Field Lines
The electric force caused by any collection of charges at rest is a conservative force. The work \( W \) done by the electric force on a charged particle moving in an electric field can be represented by the change in a potential-energy function \( U \).

\[
W_{a \to b} = U_a - U_b \quad (23.2)
\]

The electric potential energy for two point charges \( q \) and \( q_0 \) depends on their separation \( r \). The electric potential energy for a charge \( q_0 \) in the presence of a collection of charges \( q_1, q_2, q_3 \) depends on the distance from \( q_0 \) to each of these other charges. (See Examples 23.1 and 23.2)

\[
U = \frac{1}{4\pi\varepsilon_0} \frac{qq_0}{r} \quad \text{(two point charges)} \quad (23.9)
\]

\[
U = \frac{q_0}{4\pi\varepsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots \right) \quad (23.10)
\]

\[
= \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{\infty} \frac{q_i}{r_i} \quad \text{(} q_0 \text{ in presence of other point charges)} \quad (23.10)
\]

Potential, denoted by \( V \), is potential energy per unit charge. The potential difference between two points equals the amount of work that would be required to move a unit positive test charge between those points. The potential \( V \) due to a quantity of charge can be calculated by summing (if the charge is a collection of point charges) or by integrating (if the charge is a distribution). (See Examples 23.3, 23.4, 23.5, 23.7, 23.11, and 23.12)

\[
V = \frac{U}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \quad \text{(due to a point charge)} \quad (23.14)
\]

\[
V = \frac{U}{q_0} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{\infty} \frac{q_i}{r_i} \quad \text{(due to a collection of point charges)} \quad (23.15)
\]

\[
V = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r} \quad \text{(due to a charge distribution)} \quad (23.16)
\]
The potential difference between two points \( a \) and \( b \), also called the potential of \( a \) with respect to \( b \), is given by the line integral of \( \vec{E} \). The potential at a given point can be found by first finding \( \vec{E} \) and then carrying out this integral. (See Examples 23.6, 23.8, 23.9, and 23.10)

\[
V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E \cos \phi \, dl \quad (23.17)
\]

An equipotential surface is a surface on which the potential has the same value at every point. At a point where a field line crosses an equipotential surface, the two are perpendicular. When all charges are at rest, the surface of a conductor is always an equipotential surface and all points in the interior of a conductor are at the same potential. When a cavity within a conductor contains no charge, the entire cavity is an equipotential region and there is no surface charge anywhere on the surface of the cavity.
End of Chapter 23

You are responsible for the material covered in T&F Sections 23.1-23.4

You are expected to:

• Understand the following terms:
  Work, Potential Energy, Electrostatic Potential, Potential Difference, Equipotential Surface

• Understand the relationship between work and potential energy. In particular, be able to distinguish between work done “by the field” and work done by the force moving a charge in a field.

• Be able to calculate the electrostatic potential energy of simple charge geometries using either the sum rule for discreet geometries or the integral rule for continuous geometries

• CLEARLY DISTINGUISH BETWEEN POTENTIAL ENERGY AND ELECTROSTATIC POTENTIAL involved.

• Be able to calculate the electrostatic potential at a point in space due to simple charge geometries using either the sum rule for discreet geometries or the integral rule for continuous geometries

• Understand the convention of setting the zero of potential energy at “infinity”

Recommended Y&F Exercises chapter 23:

1, 6, 9, 14, 21, 23, 35