Electric Potential Energy
and
Electric Potential
Energy Considerations

When a force, $F$, acts on a particle, work is done on the particle in moving from point $a$ to point $b$

$$W_{a\rightarrow b} = \int_a^b \vec{F} \cdot d\vec{l}$$

If the force is a conservative, then the work done can be expressed in terms of a change in potential energy

$$W_{a\rightarrow b} = -(U_b - U_a) = -\Delta U$$

Also if the force is conservative, the total energy of the particle remains constant

$$KE_a + PE_a = KE_b + PE_b$$
Work Done by Uniform Electric Field

Force on charge is
\[ \vec{F} = q_0 \vec{E} \]

Work is done on the charge by field
\[ W_{a \rightarrow b} = Fd = q_0 Ed \]

The work done is *independent* of path taken from point a to point b because

The Electric Force is a *conservative force*
Electric Potential Energy

The work done by the force is the same as the change in the particle’s potential energy

\[ W_{a \rightarrow b} = -(U_b - U_a) = -\Delta U \]

\[ U_b - U_a = -\int_{a}^{b} F \cdot ds = -qE_{\text{uniform}} (y_b - y_a) \]

The work done only depends upon the change in position
Electric Potential Energy

General Points

1) Potential Energy *increases* if the particle moves in the direction *opposite* to the force on it

*Work will have to be done by an external agent for this to occur*

and

2) Potential Energy *decreases* if the particle moves in the *same* direction as the force on it
Potential Energy of Two Point Charges

Suppose we have two charges $q$ and $q_0$ separated by a distance $r$

The force between the two charges is given by Coulomb’s Law

$$F = \frac{1}{4\pi \varepsilon_0} \frac{qq_0}{r^2}$$

We now displace charge $q_0$ along a radial line from point $a$ to point $b$

The force is not constant during this displacement

$$W_{a\rightarrow b} = \int_{r_a}^{r_b} F_r \, dr = \int_{r_a}^{r_b} \frac{1}{4\pi \varepsilon_0} \frac{qq_0}{r^2} \, dr = \frac{qq_0}{4\pi \varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right)$$
Potential Energy of Two Point Charges

The work done is not dependent upon the path taken in getting from point $a$ to point $b$.

The work done is related to the component of the force along the displacement:

$$\vec{F} \cdot d\vec{r}$$
Potential Energy

Looking at the work done we notice that there is the same functional at points a and b and that we are taking the difference.

We define this functional to be the potential energy:

\[ W_{a\rightarrow b} = \frac{qq_0}{4\pi \varepsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \]

The signs of the charges are included in the calculation.

The potential energy is taken to be zero when the two charges are infinitely separated.
A System of Point Charges

Suppose we have more than two charges

*Have to be careful of the question being asked*

Two possible questions:

1) Total Potential energy of one of the charges with respect to remaining charges

or

2) Total Potential Energy of the System
Case 1: Potential Energy of one charge with respect to others

Given several charges, $q_1 \ldots q_n$, in place

Now a test charge, $q_0$, is brought into position

Work must be done against the electric fields of the original charges

This work goes into the potential energy of $q_0$

We calculate the potential energy of $q_0$ with respect to each of the other charges and then

Just sum the individual potential energies

$$PE_{q_0} = \sum_i \frac{1}{4\pi \varepsilon_0} \frac{q_0 q_i}{r_i}$$

**Remember - Potential Energy is a Scalar**
Case 2: Potential Energy of a System of Charges

Start by putting first charge in position
No work is necessary to do this

Next bring second charge into place
Now work is done by the electric field of the first charge. This work goes into the potential energy between these two charges.

Now the third charge is put into place
Work is done by the electric fields of the two previous charges. There are two potential energy terms for this step.

We continue in this manner until all the charges are in place

The total potential is then given by

\[ PE_{\text{system}} = \sum_{i<j} \frac{1}{4\pi \varepsilon_0} \frac{q_i q_j}{r_{ij}} \]
Example 1

Two test charges are brought separately to the vicinity of a positive charge $Q$

Charge $+q$ is brought to pt A, a distance $r$ from $Q$

Charge $+2q$ is brought to pt B, a distance $2r$ from $Q$

I) Compare the potential energy of $q$ ($U_A$) to that of $2q$ ($U_B$)

(a) $U_A < U_B$    (b) $U_A = U_B$    (c) $U_A > U_B$

II) Suppose charge $2q$ has mass $m$ and is released from rest from the above position (a distance $2r$ from $Q$). What is its velocity $v_f$ as it approaches $r = \infty$ ?

(a) $v_f = \sqrt{\frac{1}{4\pi\varepsilon_0} \frac{Qq}{mr}}$    (b) $v_f = \sqrt{\frac{1}{2\pi\varepsilon_0} \frac{Qq}{mr}}$    (c) $v_f = 0$
Example 2

Two test charges are brought separately to the vicinity of a positive charge $Q$

- Charge $+q$ is brought to pt A, a distance $r$ from $Q$
- Charge $+2q$ is brought to pt B, a distance $2r$ from $Q$

I) Compare the potential energy of $q$ ($U_A$) to that of $2q$ ($U_B$)

(a) $U_A < U_B$  (b) $U_A = U_B$  (c) $U_A > U_B$

The potential energy of $q$ is proportional to $Qq/r$

The potential energy of $2q$ is proportional to $Q(2q)/(2r) = Qq/r$

Therefore, the potential energies $U_A$ and $U_B$ are EQUAL!!!
Example 3

II) Suppose charge $2q$ has mass $m$ and is released from rest from the above position (a distance $2r$ from $Q$). What is its velocity $v_f$ as it approaches $r = \infty$?

\[
\begin{align*}
(a) \quad v_f &= \sqrt{\frac{1}{4\pi\varepsilon_0} \frac{Qq}{mr}} \\
(b) \quad v_f &= \sqrt{\frac{1}{2\pi\varepsilon_0} \frac{Qq}{mr}} \\
(c) \quad v_f &= 0
\end{align*}
\]

The principle at work here is **CONSERVATION OF ENERGY**.

Initially:
- The charge has no kinetic energy since it is at rest.
- The charge does have potential energy (electric) = $U_B$.

Finally:
- The charge has no potential energy ($U \propto 1/R$)
- The charge does have kinetic energy = $KE$

\[
U_B = KE \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_0} \frac{Q(2q)}{2r} = \frac{1}{2} mv_f^2 \quad \Rightarrow \quad v_f^2 = \frac{1}{2\pi\varepsilon_0} \frac{Qq}{mr}
\]
Electric Potential

Recall Case 1 from before

The potential energy of the test charge, \( q_0 \), was given by

\[
PE_{q_0} = \sum_i \frac{1}{4\pi \varepsilon_0} \frac{q_0 q_i}{r_i}
\]

Notice that there is a part of this equation that would remain the same regardless of the test charge, \( q_0 \), placed at point \( a \)

The value of the test charge can be pulled out from the summation

\[
PE_{q_0} = q_0 \sum_i \frac{1}{4\pi \varepsilon_0} \frac{q_i}{r_i}
\]
Electric Potential

We define the term to the right of the summation as the electric potential at point \( a \)

\[
\text{Potential}_a = \sum_i \frac{1}{4\pi \varepsilon_0} \frac{q_i}{r_i}
\]

Like energy, potential is a scalar

We define the potential of a given point charge as being

\[
\text{Potential} = V = \frac{1}{4\pi \varepsilon_0} \frac{q}{r}
\]

This equation has the convention that the potential is zero at infinite distance
Electric Potential

The potential at a given point

*Represents the potential energy that a positive unit charge would have, if it were placed at that point*

It has units of

$$\text{Volts} = \frac{\text{Energy}}{\text{charge}} = \frac{joules}{coulomb}$$
Electric Potential

General Points for either positive or negative charges

The Potential *increases* if you move in the direction *opposite* to the electric field

and

The Potential *decreases* if you move in the *same* direction as the electric field
Example 4

Points A, B, and C lie in a uniform electric field.

What is the potential difference between points A and B?

\[ \Delta V_{AB} = V_B - V_A \]

- a) \( \Delta V_{AB} > 0 \)
- b) \( \Delta V_{AB} = 0 \)
- c) \( \Delta V_{AB} < 0 \)

The electric field, \( E \), points in the direction of decreasing potential.

Since points A and B are in the same relative horizontal location in the electric field there is no potential difference between them.
Example 5

Points A, B, and C lie in a uniform electric field.

Point C is at a higher potential than point A.

True  False

As stated previously the electric field points in the direction of decreasing potential

Since point C is further to the right in the electric field and the electric field is pointing to the right, point C is at a lower potential

The statement is therefore false
Example 6

Points A, B, and C lie in a uniform electric field.

If a negative charge is moved from point A to point B, its electric potential energy

a) Increases.  b) decreases.  c) doesn’t change.

The potential energy of a charge at a location in an electric field is given by the product of the charge and the potential at the location.

As shown in Example 4, the potential at points A and B are the same.

Therefore the electric potential energy also doesn’t change.
Example 7

Points A, B, and C lie in a uniform electric field. Compare the potential differences between points A and C and points B and C.

a) $V_{AC} > V_{BC}$ b) $V_{AC} = V_{BC}$ c) $V_{AC} < V_{BC}$

In Example 4 we showed that the potential at points A and B were the same.

Therefore the potential difference between A and C and the potential difference between points B and C are the same.

Also remember that potential and potential energy are scalars and directions do not come into play.
Work and Potential

The work done by the electric force in moving a test charge from point $a$ to point $b$ is given by

$$W_{a\rightarrow b} = \int_{a}^{b} \vec{F} \cdot d\vec{l} = \int_{a}^{b} q_0 \vec{E} \cdot d\vec{l}$$

Dividing through by the test charge $q_0$ we have

$$V_a - V_b = \int_{a}^{b} \vec{E} \cdot d\vec{l}$$

Rearranging so the order of the subscripts is the same on both sides

$$V_b - V_a = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$
Potential

From this last result

\[ V_b - V_a = - \int_a^b \vec{E} \cdot d\vec{l} \]

We get

\[ dV = - \vec{E} \cdot d\vec{l} \text{ or } \frac{dV}{dx} = -E \]

We see that the electric field points in the direction of *decreasing* potential

We are often more interested in potential differences as this relates directly to the work done in moving a charge from one point to another
Example 8

If you want to move in a region of electric field without changing your electric potential energy. You would move

a) Parallel to the electric field
b) Perpendicular to the electric field

The work done by the electric field when a charge moves from one point to another is given by

\[ W_{a \rightarrow b} = \int_{a}^{b} \overrightarrow{F} \cdot d\overrightarrow{l} = \int_{a}^{b} q_0 \overrightarrow{E} \cdot d\overrightarrow{l} \]

The way no work is done by the electric field is if the integration path is perpendicular to the electric field giving a zero for the dot product
Example 9

A positive charge is released from rest in a region of electric field. The charge moves:

- **a)** towards a region of smaller electric potential
- **b)** along a path of constant electric potential
- **c)** towards a region of greater electric potential

A positive charge placed in an electric field will experience a force given by \( F = q \ E \)

But \( E \) is also given by \( E = -\frac{dV}{dx} \)

Therefore \( F = q \ E = -q \frac{dV}{dx} \)

Since \( q \) is positive, the force \( F \) points in the direction opposite to increasing potential or in the direction of decreasing potential.
Units for Energy

There is an additional unit that is used for energy in addition to that of joules.

A particle having the charge of \( e \) \((1.6 \times 10^{-19} \text{ C})\) that is moved through a potential difference of 1 Volt has an increase in energy that is given by

\[
W = qV = 1.6 \times 10^{-19} \text{ joules}
\]

\[
= 1 \text{ eV}
\]
 Equipotential Surfaces

It is possible to move a test charge from one point to another without having any net work done on the charge.

This occurs when the beginning and end points have the same potential.

It is possible to map out such points and a given set of points at the same potential form an **equipotential surface**.
Equipotential Surfaces

Examples of equipotential surfaces

Point Charge

Two Positive Charges
Equipotential Surfaces

The electric field does no work as a charge is moved along an equipotential surface.

Since no work is done, there is no force, \( qE \), along the direction of motion.

The electric field is perpendicular to the equipotential surface.
What about Conductors

In a static situation, the surface of a conductor is an equipotential surface.

But what is the potential inside the conductor if there is a surface charge?

We know that $E = 0$ inside the conductor.

This leads to

$$\frac{dV}{dx} = 0 \text{ or } V = \text{constant}$$
What about Conductors

The value of the potential inside the conductor is chosen to match that at the surface.
Potential Gradient

The equation that relates the derivative of the potential to the electric field that we had before

\[ \frac{dV}{dx} = -E \]

can be expanded into three dimensions

\[ \vec{E} = -\vec{\nabla}V \]

\[ \vec{E} = -\left( i \frac{dV}{dx} + j \frac{dV}{dy} + k \frac{dV}{dz} \right) \]
Potential Gradient

For the gradient operator, use the one that is appropriate to the coordinate system that is being used.
Example 10

The electric potential in a region of space is given by

\[ V(x) = 3x^2 - x^3 \]

The x-component of the electric field \( E_x \) at \( x = 2 \) is

(a) \( E_x = 0 \) \hspace{1cm} (b) \( E_x > 0 \) \hspace{1cm} (c) \( E_x < 0 \)

We know \( V(x) \) “everywhere”

To obtain \( E_x \) “everywhere”, use

\[ \vec{E} = -\nabla V \implies E_x = -\frac{dV}{dx} \implies E_x = -6x + 3x^2 \]

\[ E_x(2) = -6(2) + 3(2)^2 = 0 \]