The Vacuum Canon Equation

At the simplest level, one merely assumes that a constant force on the ping-pong ball (area A, mass m) due to air pressure (P) accelerates it the down the length of the tube. Straight-forward dynamics gives the velocity as a function of distance x down the tube as

$$v = \sqrt{\frac{2PAx}{m}}\ .$$

Our ping-pong balls have a mass of 2.3 g and a diameter of 38 mm. Using $P = 10^5$ Pa and $x = 1.83$ m (6 ft) you get the result $v = 425$ m/s –faster than the speed of sound. Apparently you can increase the muzzle velocity without limit by using longer and longer tubes!

As mentioned on the vacuum-canon web page, actual speeds on the order of 300 m/s have been measured. This is still incredible, but more believable.

The major omission in the analysis above is that a considerable amount of air must also be accelerated. Using the density of air as 1.3 kg/m^3, one can calculate that a section of the tube 1.57 m long contains the same mass as our ping-pong ball. Let's refigure the problem including this inrushing air:

At our (new) simplest-level-of-approximation we assume air is incompressible with density $\rho = 1.3$ kg/m^3. For the ball a distance x down the tube, the force PA on the end accelerates a mass m+M, where M is the mass of air filling the tube to point x. Newton's law then gives us

$$\frac{d}{dt}(m + M)v = PA.$$

Since $M = \rho Ax$, we get

$$v^2 + \left(\frac{m}{\rho A} + x\right)\dot{v} = \frac{P}{\rho}.$$

We can immediately see that there is a limiting, or terminal, velocity v_T when $\dot{v} = 0$:

$$v_T = \sqrt{\frac{P}{\rho}} = 277 \text{ m/s.}$$

This is in pretty good agreement with the cited measurements, especially when we have treated the air motion so simplistically.

If there is a terminal velocity, how long a canon do we need? (An important question for the Demo Boy.) To solve the D.E. we write

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{dx} v \, .$$

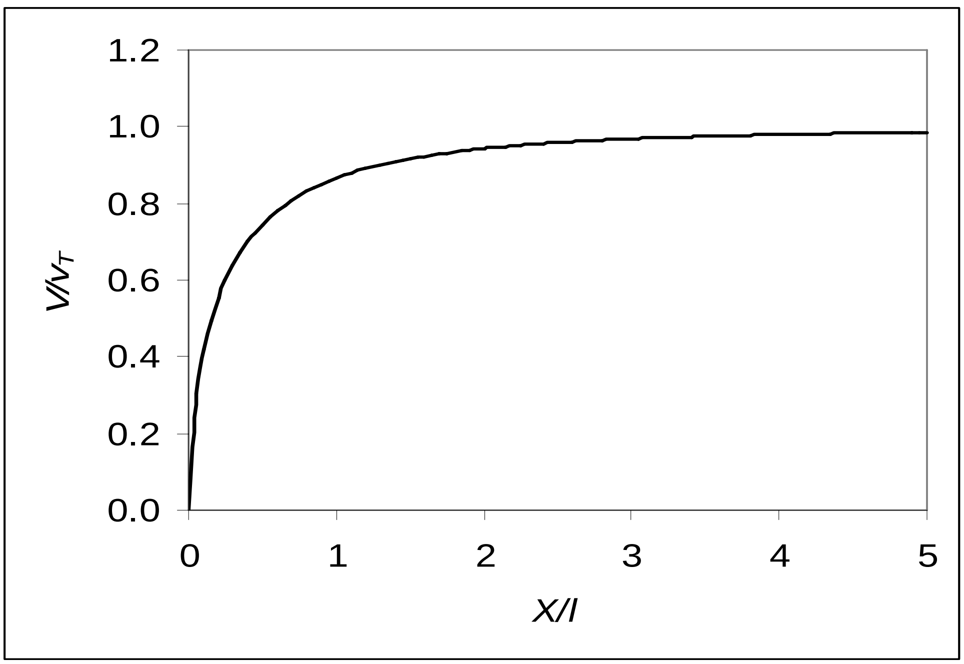
The equation is then separable and we integrate

$$\int_0^V \frac{v dv}{v_T^2 - v^2} = \int_0^X \frac{dx}{l + x} \, ,$$

where the length $l \equiv m / \rho A = 1.57 \text{ m}$ is again the characteristic distance for which the projectile mass equals the enclosed air mass. One gets

$$\frac{V}{v_T} = \sqrt{1 - \frac{1}{(1 + X/l)^2}} \, .$$

A plot shows the asymptotic behavior of *V*(*X*):



So, in this model we don't need a very long canon; a length of $X = l = 1.6$ m gives us a muzzle velocity 87% of the maximum possible. Doubling to $2l = 3.2$ m would increase the muzzle velocity only to 94% of v_T. (Our newest 6-ft canon has

$v_{muzzle} = 0.89 v_T$.) This explains why we saw no real difference in the canon's performance when it was cut down for repair.

Now that we have an expression for $v(x)$, we can also find the acceleration of the ball at any point in the tube:

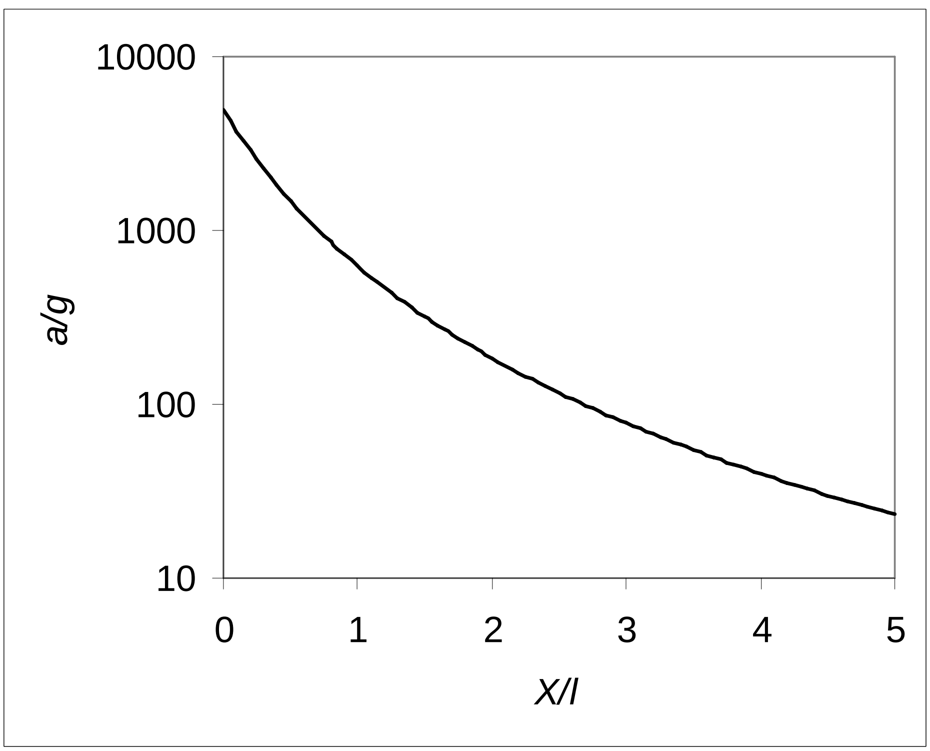
$$a = v\frac{dv}{dx} = \frac{v_T^2}{l}\frac{1}{(1+x/l)^3}.$$

So, there is a maximum acceleration at $x = 0$ of

$$a_{\max} = \frac{v_T^2}{l} = 4.9\times 10^4 \text{ m/s}^2,$$

almost 5000 g's! As one might expect, this is the same as the (constant) acceleration found in the simpler calculation ($a = PA/m$). Now the acceleration decreases as the mass inside the tube increases.

A plot of *a/g*:



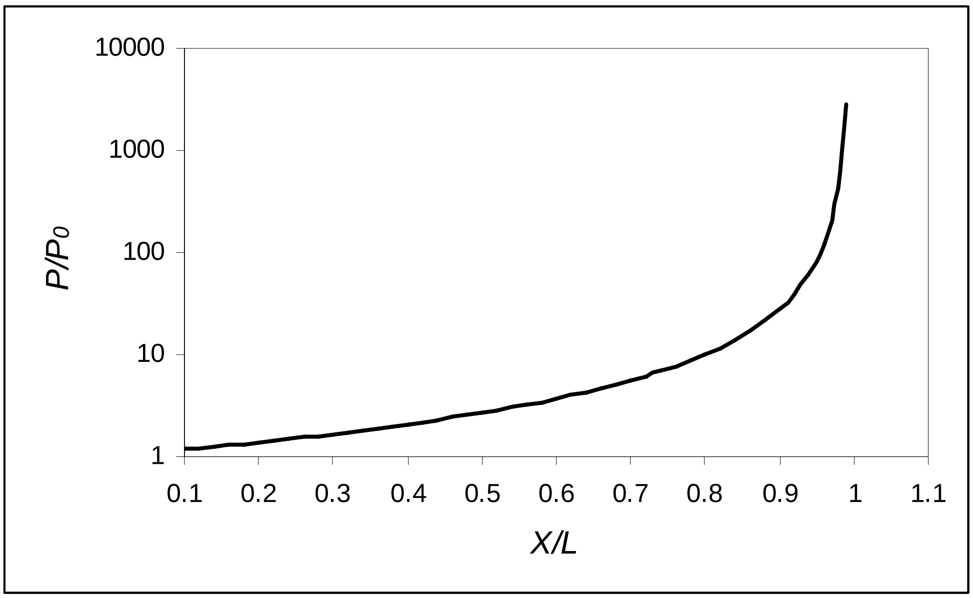
(Many thanks to Stu Elston for his ideas on this).

Compression effects (Just for fun)

One might also worry about the compression of the residual air in front of the ping pong ball: Once this pressure becomes comparable to the ambient air pressure acceleration would cease. Assuming adiabatic compression, the pressure in front of the ball is

$$\frac{P(x)}{P_0} \approx \frac{1}{(1 - x/L - 2r/3L)^{7/5}},$$

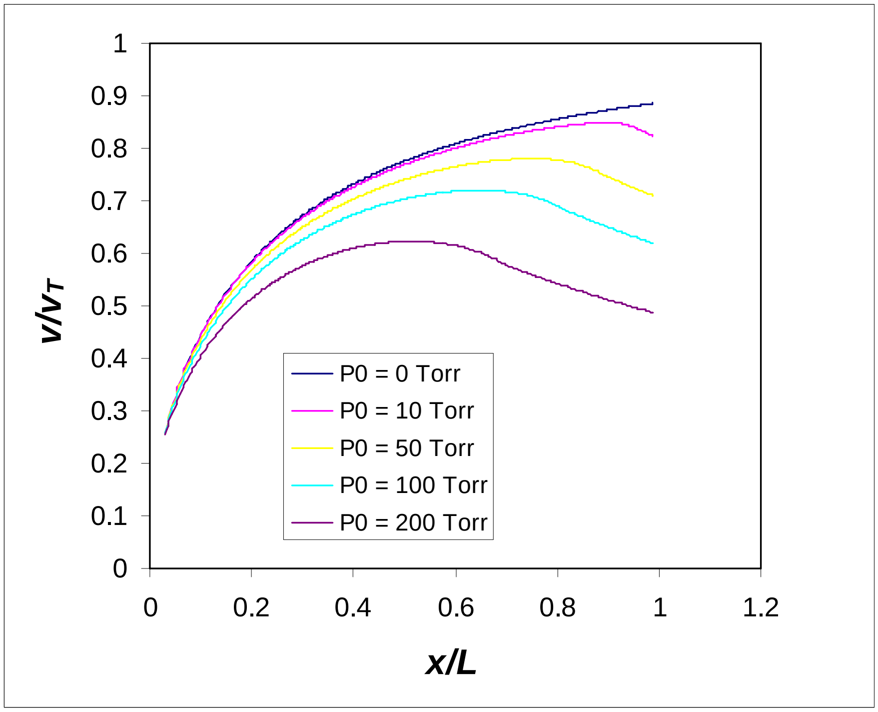
where P_0 is the pump-down pressure, L is the length of the tube, and r is the ping pong ball's radius. (The $7/5^{th}$ power arises from using γ for diatomic N_2.) The maximum distance the ball travels down the tube is $x_{max} = L-r$. For our new canon, L=1.83m and r=19mm. The compression ratio looks like:

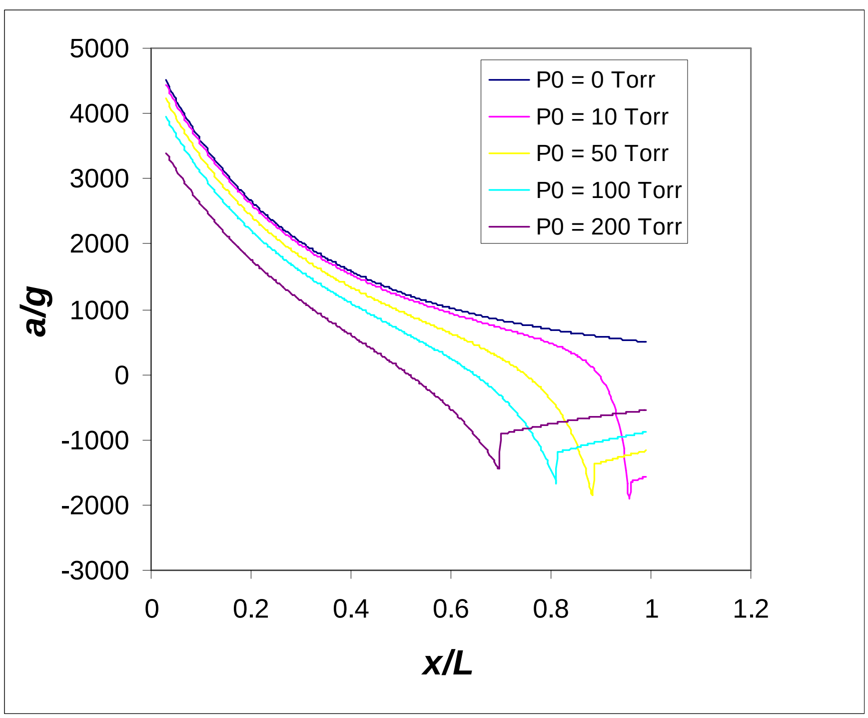


There would seem to be only a small retarding force for most the distance down the tube: for a distance of 80% down the tube the compression ratio is only 10. However in the last little bit the pressure could become quite high – implying that the tape sealing the barrel is actually blow off before the ball hits.

This estimate also ignores the volume of the pump-down hose if vacuum pump is connected to the muzzle end. None-the-less, there is a noticeable lack of can-crushing power unless the canon is pumped down to around P_0 = 2 Torr.

When the retarding force $-P(x)A$ is included in the problem, the differential equation is no longer separable. It can be solved numerically with the following results:





The kinks in the acceleration curves come about from the bursting end window: I've arbitrarily set a bursting limit of 1.2 atm for $P(x)$ –if exceeded the retarding pressure is set back to atmospheric. Probably this should be set higher, but the point of the calculation is that the point of maximum velocity ($a=0$) occurs before bursting.