Generator Coordinate Method

Aim of the Generator Coordinate Method (GCM):

1. to include correlations (beyond antisymmetry) that arise in transitional nuclei and to treat anharmonic effects
2. to restore symmetries

Let $a = (a_1, a_2, \ldots, a_N)$ be N parameters (called "coordinates") and let $|\Phi(a)\rangle$ be Slater-determinants parametrized by a. We want to approximate the ground state of the nucleus of interest as

$$|\Psi\rangle = \int da\, f(a)|\Phi(a)\rangle, \tag{1}$$

and want to determine the *optimal* weight function $f(a)$.

Examples:

1. $a = \beta$ deformation parameter. For transitional nuclei or spontaneous fission.
2. $|\Phi(z)\rangle = \exp\left(\Sigma_{j<k}\, z_{jk} a_j^\dagger a_k^\dagger\right)|\phi(0)\rangle$ yields Slater determinant (Thouless theorem).

Variation of energy $E = \langle\Psi|H|\Psi\rangle / \langle\Psi|\Psi\rangle$ yields *Hill-Wheeler equation* that determines the weight function:

$$\int da' \langle\Phi(a)|H-E|\Phi(a')\rangle f(a') = 0. \tag{2}$$

This is a generalized eigenvalue problem of the structure

$$\mathcal{H}f = Ef, \tag{3}$$

where

$$\begin{aligned} \mathcal{H}(a,a') &= \langle\Phi(a)|H|\Phi(a')\rangle \quad \text{and} \\ \mathcal{N}(a,a') &= \langle\Phi(a)|\Phi(a')\rangle \end{aligned}$$

are the Hamiltonian and the overlap matrix, respectively.

Solution of Hill-Wheeler equation

1. Diagonalize overlap matrix $\int da' \mathcal{N}(a,a') u_k(a') = n_k u_k(a)$. Note: Overlap matrix is non-negative by definition.
2. Choose those sets of norms n_k and eigenstates $u_k(a)$ that are numerically nonzero (i.e. larger than machine epsilon).
3. Compute $H_{kl} = \int da\, da'\, u_k^*(a)\mathcal{H}(a,a')u_l(a')/\sqrt{n_k n_l}$.
4. Solve eigenvalue problem $\sum_l H_{kl} g_l = E g_k$.
5. Construct weight function $f(a) = \sum_k g_k u_k(a)/\sqrt{n_k}$.

Note: Projection methods for symmetry restoration are of GCM type:
(i) Axial symmetry projection:

$$|\Psi(J_z=0)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i\phi \hat{J}_z}|\Psi(\phi)\rangle$$

(ii) Particle number projection of BCS state:

$$|\Psi(N)\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} d\phi e^{i\phi(\hat{N}-N)}|BCS\rangle$$

(iii) General: Let χ be the character of the symmetry group G whose irreducible representation is sought:

$$|\Psi(\chi)\rangle = \frac{1}{|G|}\int d\phi\chi(\phi)\hat{R}(\phi)|\Phi\rangle.$$

The integral is over the group, weighted by the character χ, and the operator $\hat{R}$ is the group operation (e.g. rotation in case of $G = SO(3)$).

Giant Dipole Resonances

Properties of the giant dipole resonance (GDR):

1. $J^\pi = 1^-$
2. dominates the cross section
3. exhausts sum rule
4. centroid $E_{1^-} \approx 80A^{-1/3}$ MeV, width $\Gamma_{1^-} \approx$ a few MeV.

Total cross section for electric dipole excitations from ground state

$$\sigma_{Total} = \frac{4\pi^2 e^2}{\hbar c} S_1(D),$$

where

$$\begin{aligned} S_1(D) &\equiv \sum_\nu (E_\nu - E_0)|\langle \nu|D|0\rangle|^2 \\ &= \frac{1}{2}\langle 0|[D,[H,D]]|0\rangle \end{aligned}$$

is the energy weighted sum rule of the dipole operator

$$\vec{D} = \frac{NZ}{A}\left(\frac{1}{Z}\sum_p \vec{r}_p - \frac{1}{N}\sum_n \vec{r}_n\right).$$

If the Hamiltonian H consists of kinetic energy and velocity-independent potentials, and the operator D is one-body, e.g. $D = \Sigma_{j=1}^{A} d(\vec{r}_j)$ one finds

$$S_1(D) = \frac{\hbar^2}{2m} A \langle 0|(\nabla d)^2|0\rangle.$$

For the dipole operator: $S_1(D) = \frac{NZ}{A}\frac{\hbar^2}{2m}$, and the total dipole cross section becomes

$$\sigma_{Total} = \frac{2\pi^2 e^2 \hbar}{mc} \approx 0.06 \frac{NZ}{A} [\text{Mev barn}].$$

Since $\sigma(GDR) \approx 0.5\sigma_{Total}$, one has approximatively

$$\begin{aligned} |GDR\rangle &= D_z|0\rangle \\ &= \frac{NZ}{A}\left(\frac{1}{Z}\sum_{p,p'}\langle p|z|p'\rangle a_p^\dagger a_{p'} - \frac{1}{N}\sum_{n,n'}\langle n|z|n'\rangle a_n^\dagger a_{n'}\right)|0\rangle \end{aligned} \quad (4)$$

Goldhaber and Teller (1948): GDR is oscillations of proton center-of-mass versus neutron center-of-mass.
The Equation (4) also shows that the GDR is based on $1p$-$1h$ excitations from the ground state.

Other giant resonances:

1. Giant quadrupole resonance: $E_{2^+} \approx 60 - 65\text{MeV}A^{-1/3}$
2. Giant monopole resonance: $E_{0^+} \approx 80\text{MeV}A^{-1/3}$ (breathing mode)
3. Giant $M1$ resonance: Scissors mode of protons vs. neutrons