The Fermionic Fock Space

Vladimir Aleksandrovich Fock (1898–1974) was a Soviet physicist, who did foundational work on quantum mechanics.

His primary scientific contribution lies in the development of quantum physics, although he also contributed significantly to the fields of mechanics, theoretical optics, theory of gravitation, physics of continuous medium. In 1926 he generalized the Klein-Gordon equation. He gave his name to Fock space [Z. Phys.,75 (1932) 622], the Fock representation and Fock state, and developed the Hartree-Fock method in 1930.

Fock was a professor at Saint Petersburg State University

Many-fermion operators and permutational symmetry

Let us start from the Hamiltonian operator:

$$\hat{H} = \hat{T} + \hat{V}_1 + \hat{V}_2 + \cdots$$

Kinetic energy term ($\hat{T}$); Interaction with external field ($\hat{V}_1$); Two-body (pairwise) interaction ($\hat{V}_2$)

$$\hat{T} = \sum_{i=1}^{A} \frac{\boldsymbol{p}_i^2}{2m_i}$$

$$\hat{V}_1 = \sum_{i=1}^{A} v_1(x_i)$$

one-particle operators (sums of contributions from individual particles)

single-particle degrees of freedom

$$x_i \equiv (\boldsymbol{r}_i, \sigma_i, \tau_i)$$

$$\hat{V}_2 = \sum_{i<j}^{A} v_2(x_i, x_j) = \frac{1}{2} \sum_{i \neq j}^{A} v_2(x_i, x_j)$$

$$v_2(x, y) = v_2(y, x)$$

$v_2(x, x)$ describes self-interaction. If such a term exists, it can always be included in V_1

Three-body interaction:

$$\hat{V}_3 = \sum_{i<j<k}^{A} v_3(x_i, x_j, x_k) = \frac{1}{6} \sum_{i \neq j \neq k}^{A} v_3(x_i, x_j, x_k)$$

Exchange operators

$\hat{\mathcal{P}}_{ij}$ - exchanges particles *i* and *j*

$$\hat{\mathcal{P}}_{ij}\Psi\left(x_1\cdots x_i\cdots x_j\cdots x_A\right)=\Psi\left(x_1\cdots x_j\cdots x_i\cdots x_A\right)$$

$\hat{\mathcal{P}}_{ij}$ is hermitian and unitary:

$$\hat{\mathcal{P}}_{ij}^{+}=\hat{\mathcal{P}}_{ij},\quad \hat{\mathcal{P}}_{ij}^{2}=1$$

→ eigenvalues of $\hat{\mathcal{P}}_{ij}$ are ± 1

(identical particles cannot be distinguished)

For identical particles, measurements performed on quantum states Ψ and $\hat{\mathcal{P}}_{ij}\Psi$ have to yield identical results

A principle, supported by experiment

This principle implies that all mmany-body wave functions are eigenstates of $\hat{\mathcal{P}}_{ij}$

$$\hat{\mathcal{P}}_{ij}\Psi=p_{ij}\Psi,\quad p_{ij}=\pm 1$$

→ is a basis of one-dimensional representation of the permutation group

There are only two one-dimensional representations of the permutation group:

$p_{ij}=+1$ for all *i,j* - fully symmetric representation

$p_{ij}=-1$ for all *i,j* - fully antisymmetric representation

Consequently, systems of identical particles form two separate classes:

$\hat{\mathcal{P}}_{ij}\Psi = \Psi$ bosons (integer spins)

$\hat{\mathcal{P}}_{ij}\Psi = -\Psi$ fermions (half-integer spins)

For spin-statistics theorem, see W. Pauli, Phys. Rev. 58, 716-722(1940)

$$\left[\hat{\mathcal{P}}_{ij}, \hat{H}\right] = 0$$

Many-body space

Fock space

From Wikipedia, the free encyclopedia

The **Fock space** is an algebraic system (Hilbert space) used in quantum mechanics to describe quantum states with a variable or unknown number of particles. It is named for V. A. Fock.

Technically, the Fock space is the Hilbert space made from the direct sum of tensor products of single-particle Hilbert spaces:

$$F_\nu(H) = \bigoplus_{n=0}^{\infty} S_\nu H^{\otimes n}$$

where S_ν is the operator which symmetrizes or antisymmetrizes the space, depending on whether the Hilbert space describes particles obeying bosonic ($\nu = +$) or fermionic ($\nu = -$) statistics respectively. H is the single particle Hilbert space. It describes the quantum states for a single *particle*, and to describe the quantum states of systems with n particles, or superpositions of such states, one must use a larger Hilbert space, the Fock space, which contains states for unlimited and variable number of particles. Fock states are the natural basis of this space. (See also the Slater determinant.)

One-particle (single-particle, s.p.) basis

$$\phi_1(x), \phi_2(x), \cdots \phi_M(x)$$

Dimension of the s.p. space (can be infinite)

$$\langle \phi_\mu | \phi_\nu \rangle = \int \phi_\mu^*(x) \phi_\nu(x) dx = \delta_{\mu\nu}$$

The corresponding Hilbert space is called $\mathcal{H}_1$ one particle Hilbert space.

On the level of $\mathcal{H}_1$ it is irrelevant whether one is dealing with fermions or bosons.
However, this is not the case for a two-particle space

Two-particle space

$\mathcal{H}_2 \subset \mathcal{H}_1 \otimes \mathcal{H}_1$ tensor product, has a product basis

$$\phi_\mu(x_1)\phi_\nu(x_2)$$

Is this wave function a good candidate for a two-particle state?

NO!

… but

$$\frac{1}{\sqrt{2}}\left[\phi_\mu(x_1)\phi_\nu(x_2) - \phi_\mu(x_2)\phi_\nu(x_1)\right]$$

is a basis in $\mathcal{H}_2$

The above state can also be written as:

$$\frac{1}{\sqrt{2!}}\sum_p (-1)^p \phi_\mu(x_{i_1})\phi_\nu(x_{i_2})$$

labels all permutations

There are $\binom{M}{2}$ such states!

How to generalize this result for the *A*-particle case?

A-particle space

$$\mathcal{H}_A \subset \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{A\text{-times}}$$

$$\phi_{\mu_1 \cdots \mu_A} = \frac{1}{\sqrt{A!}} \sum_p (-1)^p \phi_{\mu_1}(x_{i_1}) \cdots \phi_{\mu_A}(x_{i_A})$$

p - permutation of *A* elements:

$$p(1, 2, \cdots, A) = (i_1, i_2, \cdots, i_A)$$

a Slater determinant spanned by one-particle states!

There are $\binom{M}{A}$ such states!

There is only one state for *A*=*M* (why?)

In addition to 1-, 2-, M-particle space, there also exists a vacuum space, $\mathcal{H}_0$ containing only one state, a vacuum. One cannot associate a wave function with the vacuum space (a function of no arguments?)

Fock space

A Hilbert space being a sum of all many-body spaces:

$$\mathcal{H} \equiv \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_M$$

One can now talk about wave functions which do not have a fixed number of particles!

Dimension: $\binom{M}{0} + \binom{M}{1} + \binom{M}{2} + \ldots \binom{M}{M} = 2^M$