Hartree-Fock method, Lipkin Model (half-filled case)

Let us consider the particle-number conserving product state written in a Thouless's representation:

$$|\tilde{Z}\rangle = \exp\left(\sum_{ph} \tilde{Z}^*_{ph} a_p^+ a_h\right) a_1^+ \dots a_A^+ |0\rangle$$

In the case of the Lipkin model, the independent-particle state corresponds to the lower level (σ=-1) completely occupied:

$$a_1^+ \dots a_A^+ |0\rangle = |\frac{\Omega}{2}, -\frac{\Omega}{2}\rangle$$

In this case,

$$\sum_{ph} \tilde{Z}^*_{ph} a_p^+ a_h = \sum_m \tilde{Z}^*_{m+;m-} a_{m+}^+ a_{m-}$$

However, since all the σ=-1 and σ=+1 pairs are equivalent, one can write

$$\tilde{Z}^*_{m+;m-} = \tau \quad \text{for all } m$$

and the product state becomes a SU(2) coherent state:

$$|\tau\rangle = \frac{1}{(1+|\tau|^2)^{\Omega/2}} e^{\tau \hat{K}_+} |\Omega/2, -\Omega/2\rangle$$

In the next step, we calculate the expectation value of the Lipkin model hamiltonian in the trial state. Here, we use the method of generating functions of SU(2) coherent states and the differential calculus.

$$E_{HF} = \langle\tau|\hat{H}_{LM}|\tau\rangle_{min}$$

$$\begin{aligned} E(\tau) &= \langle\tau|\hat{H}_{LM}|\tau\rangle = \\ &= \varepsilon\langle\tau|\hat{K}_0|\tau\rangle - \frac{1}{2}V\langle\tau|\hat{K}_+^2 + \hat{K}_-^2|\tau\rangle \end{aligned}$$

Let us calculate the single-particle term first:

$$\langle\tau|\hat{K}_0|\tau\rangle = \frac{\partial}{\partial\alpha_0}\langle\tau|e^{\alpha_0\hat{K}_0}|\tau\rangle\Big|_{\alpha_0=0}$$

$$\begin{aligned} &\langle\zeta|e^{\alpha_-\hat{J}_-}e^{\alpha_0\hat{J}_0}e^{\alpha_+\hat{J}_+}|\zeta\rangle = \\ &\quad = \frac{1}{(1+|\tau|^2)^{2J}}\left[e^{-\frac{1}{2}\alpha_0} + e^{\frac{1}{2}\alpha_0}(\tau^* + \alpha_-)(\tau + \alpha_+)\right]^{2J} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha_0}\langle\tau|e^{\alpha_0\hat{K}_0}|\tau\rangle &= \frac{\Omega}{(1+|\tau|^2)^{\Omega}}\left[e^{-\frac{1}{2}\alpha_0} + e^{\frac{1}{2}\alpha_0}|\tau|^2\right]^{\Omega-1} \times \\ &\times \left(-\frac{1}{2}e^{-\frac{1}{2}\alpha_0} + \frac{1}{2}e^{\frac{1}{2}\alpha_0}|\tau|^2\right) \end{aligned}$$

$$\langle\tau|\hat{K}_0|\tau\rangle = \frac{1}{2}\Omega\frac{|\tau|^2 - 1}{|\tau|^2 + 1} = -\frac{1}{2}\Omega\cos\theta$$

$$\tau = \tan\frac{\theta}{2}e^{-i\phi}$$

$$\langle\tau|\hat{K}_+^2+\hat{K}_-^2|\tau\rangle=?$$

$$\frac{\partial}{\partial\alpha_+}\langle\tau|e^{\alpha_+\hat{K}_+}|\tau\rangle=\frac{\Omega}{(1+|\tau|^2)^\Omega}\left(1+|\tau|^2+\tau^*\alpha_+\right)^{\Omega-1}\tau^*$$

$$\frac{\partial^2}{\partial^2\alpha_+}\langle\tau|e^{\alpha_+\hat{K}_+}|\tau\rangle=\frac{\Omega(\Omega-1)}{(1+|\tau|^2)^\Omega}\left(1+|\tau|^2+\tau^*\alpha_+\right)^{\Omega-2}(\tau^*)^2$$

For $\alpha_+=0$, this gives

$$\begin{aligned}\langle\tau|\hat{K}_+|\tau\rangle &= \Omega\frac{\tau^*}{|\tau|^2+1}\\ \langle\tau|\hat{K}_+^2|\tau\rangle &= \Omega(\Omega-1)\frac{(\tau^*)^2}{(|\tau|^2+1)^2}\end{aligned}$$

This yields:

$$\begin{aligned}\langle\tau|\hat{K}_+^2+\hat{K}_-^2|\tau\rangle &= \Omega(\Omega-1)\frac{\tau^2+(\tau^*)^2}{(|\tau|^2+1)^2}\\ &= \frac{1}{2}\Omega(\Omega-1)\cos 2\phi\sin^2\theta\end{aligned}$$

Now we are ready to write down the HF energy of the Lipkin Model:

$$E(\tau) = -\frac{\varepsilon}{2}\Omega\left(\cos\theta + \frac{\chi}{2}\sin^2\theta\cos 2\phi\right)$$

where the dimensionless coupling strength is defined as

$$\chi \equiv \frac{V}{\varepsilon}(\Omega - 1)$$

Now we need to minimize *E(τ)*

$$\frac{\partial E}{\partial \theta} = 0 \quad \Rightarrow \quad \sin\theta(\chi\cos\theta\cos 2\phi - 1) = 0$$

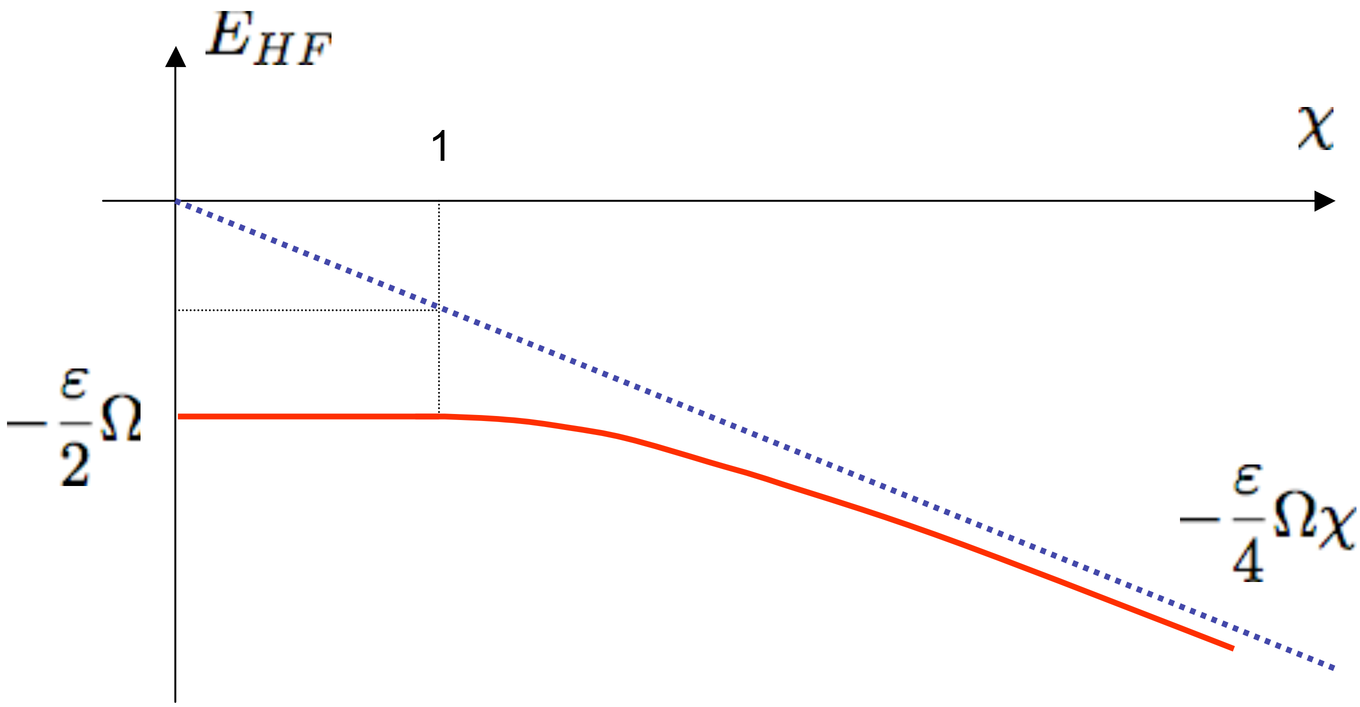
$$\frac{\partial E}{\partial \phi} = 0 \quad \Rightarrow \quad \sin^2\theta\sin 2\phi = 0$$

$$\phi_{HF} = 0$$

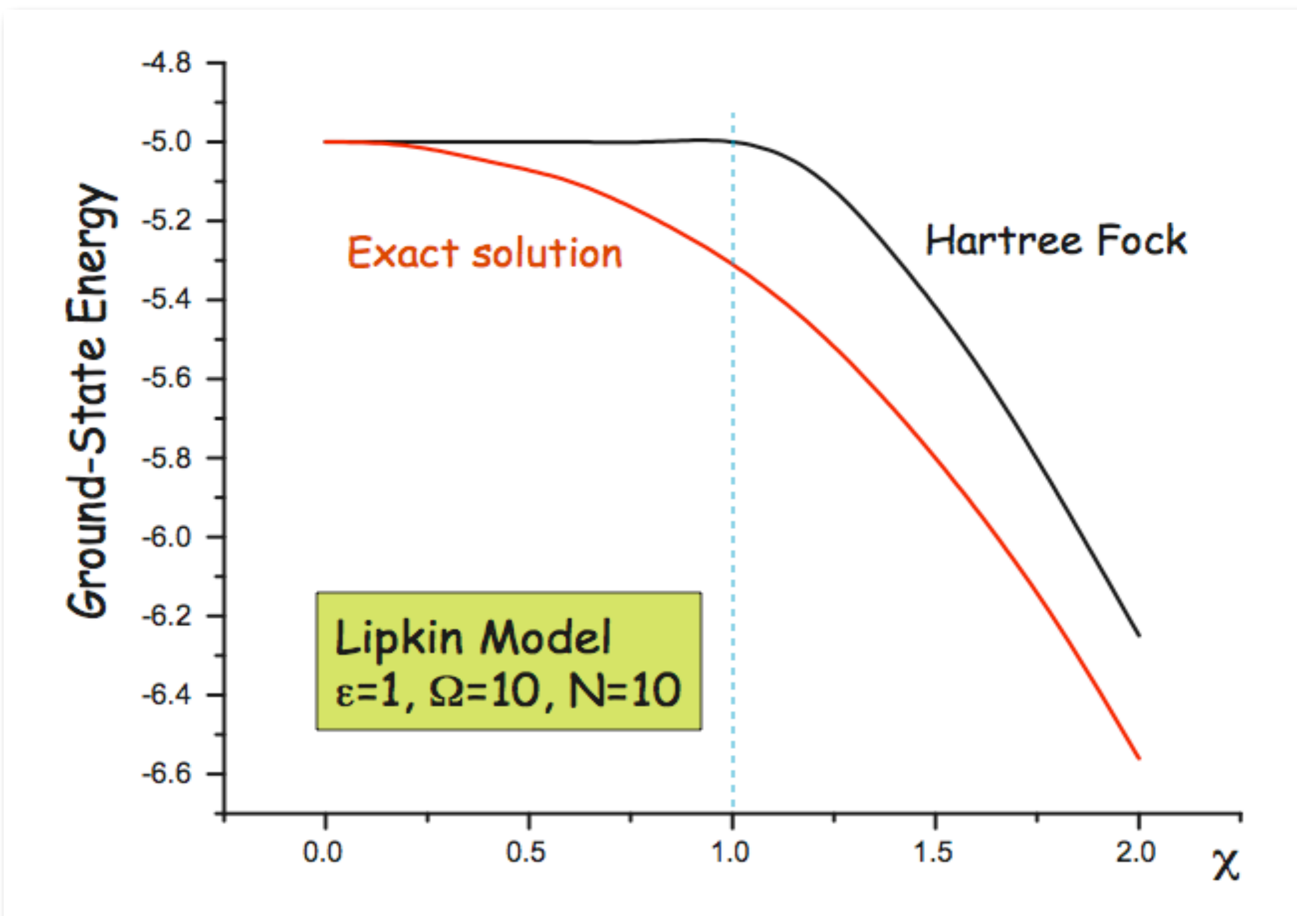
$$\sin\theta_{HF} = 0 \quad \Rightarrow \quad E_{HF} = -\frac{\Omega}{2}\varepsilon$$

$$\cos\theta_{HF} = \frac{1}{\chi} \quad \Rightarrow \quad E_{HF} = -\frac{\Omega}{4}\varepsilon\left(\chi + \frac{1}{\chi}\right)$$

$\chi \geq 1$



E_{HF}
χ
1
$-\frac{\varepsilon}{2}\Omega$
$-\frac{\varepsilon}{4}\Omega\chi$



Ground-State Energy
-4.8
-5.0
-5.2
-5.4
-5.6
-5.8
-6.0
-6.2
-6.4
-6.6
0.0
0.5
1.0
1.5
2.0
χ
Exact solution
Hartree Fock
Lipkin Model
ε=1, Ω=10, N=10

What is the curvature of the HF energy at the HF solution?

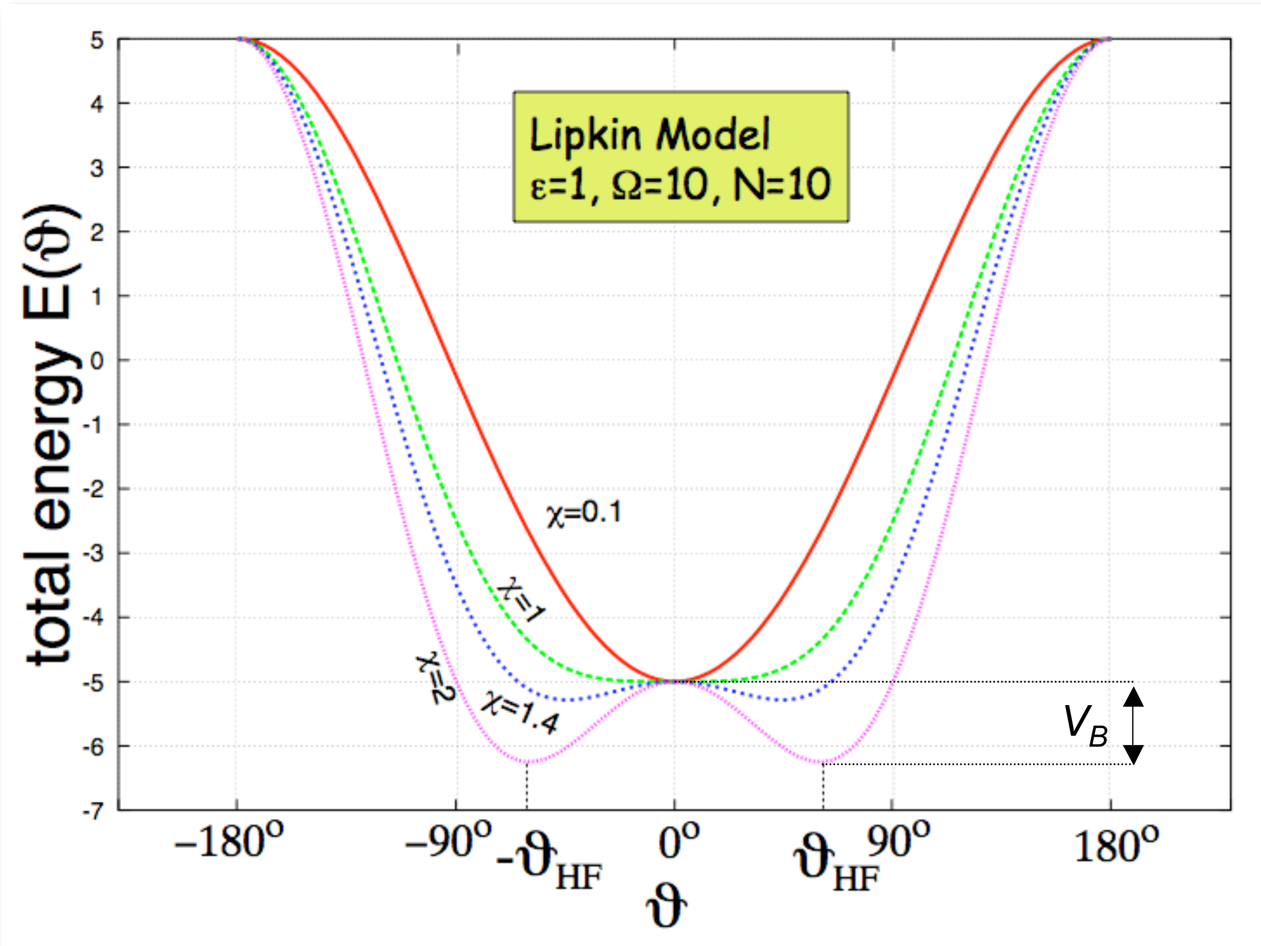
$$\frac{\partial^2 E}{\partial \theta^2} = -\frac{1}{2}\varepsilon\Omega(-\cos\theta + \chi\cos 2\theta)$$

Minimum of energy only for χ<1

$$\theta = 0 \quad \Rightarrow \quad \frac{\partial^2 E}{\partial \theta^2} = -\frac{1}{2}\varepsilon\Omega(\chi - 1)$$

$$\theta = \cos^{-1}\chi^{-1} \quad \Rightarrow \quad \frac{\partial^2 E}{\partial \theta^2} = \frac{1}{2}\varepsilon\Omega(\chi - \frac{1}{\chi}) > 0$$

Always minimum of energy for χ>1



$$E_{HF} = \begin{cases} -\frac{\varepsilon}{2}\Omega & \text{for} \quad \chi < 1 \\ -\frac{\varepsilon}{4}\Omega\left(\chi + \frac{1}{\chi}\right) & \text{for} \quad \chi \geq 1 \end{cases}$$

Barrier height: $$V_B = \frac{\varepsilon}{4}\Omega\left(\chi + \frac{1}{\chi} - 2\right)$$