

Bogoliubov Transformation, Group Structure

Any F -even unitary matrix in $2M$ will correspond to some Bogoliubov transformation. The set of such matrices forms a group (a subgroup of the unitary group of degree $2M$)

$$\mathcal{F}A_1A_2\mathcal{F} = (\mathcal{F}A_1\mathcal{F})(\mathcal{F}A_2\mathcal{F}) = \mathcal{A}_1^*\mathcal{A}_2^* = (\mathcal{A}_1\mathcal{A}_2)^*$$

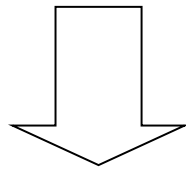
Let us now make a unitary rotation of all matrices A and define:

$$B \equiv UAU^+$$

with

$$U = \sqrt{\frac{i}{2}} \begin{pmatrix} 1 & , & -i \\ i & , & -1 \end{pmatrix}$$

$$\begin{aligned} U^+U &= 1, \\ U^T U &= -UU^T = \mathcal{F} \end{aligned}$$



$$\begin{aligned} B^* &= B \\ B^T B &= 1 \end{aligned}$$

Consequently, matrices B are real and orthogonal! They form the $O(2M)$ group - orthogonal group in $2M$

$O(2M)$ can be decomposed into $SO(2M)$ (subgroup of orthogonal matrices with $\text{Det}=+1$; proper orthogonal matrices) and a subset of orthogonal matrices with $\text{Det}=-1$ (improper orthogonal matrices). The corresponding Bogoliubov transformations will be called proper and improper, respectively

Any element of $SO(2M)$ can be represented as

$$\mathcal{B} = \exp(\mathcal{G})$$

where G is real and antisymmetric:

$$\mathcal{G}^T = -\mathcal{G}$$

$$\mathcal{G}^* = \mathcal{G}$$

Of course, $\text{Det}(\mathcal{B}) = \exp[\text{Tr}(\mathcal{G})] = 1$

Any improper matrix can be expressed as a product of a proper matrix and one (selected) improper matrix.

Proper Bogoliubov Transformations

Any proper Bogoliubov matrix transformation can be represented by

$$\mathcal{A} = \exp(i\mathcal{Y})$$

where

$$\mathcal{Y}^+ = \mathcal{Y}$$

$$\mathcal{F}\mathcal{Y}\mathcal{F} = -\mathcal{Y}^*$$

Note that Y is F -odd

$$\mathcal{A} = \mathcal{U}^+ \mathcal{B} \mathcal{U} \quad \mathcal{Y} = -i\mathcal{U}^+ \mathcal{G} \mathcal{U} \quad \mathcal{F}\mathcal{Y}\mathcal{F} = -\mathcal{Y}^*$$

Bogoliubov Transformations in the Fock space

A hermitian and F -odd matrix in $2M$ can be represented by an operator acting in the Fock space:

$$\hat{Y} = \frac{1}{2} \sum_{kl} \mathcal{Y}_{kl} c_k^+ c_l$$

Hermitian operator
in the Fock space

Looks like one-body
but is not. Why?

Expressing Y explicitly in terms of a^+ and a , one can write

$$\mathcal{Y} = \begin{pmatrix} Y & , & -X^* \\ X & , & -Y^* \end{pmatrix}$$

where

$$\begin{aligned} Y^+ &= Y \\ X^T &= -X \end{aligned}$$

$$\hat{Y} = \sum_{\mu\nu} \left(\frac{1}{2} X_{\mu\nu}^+ a_\mu^+ a_\nu^+ + Y_{\mu\nu} a_\mu^+ a_\nu + \frac{1}{2} X_{\mu\nu} a_\mu a_\nu \right) - \frac{1}{2} \text{Tr}(Y)$$

The most general hermitian operator involving all pairs of creation and annihilation operators

For any proper matrix Bogoliubov transformation

$$\mathcal{A} = \exp(i\mathcal{Y})$$

one can find a Bogoliubov transformation in the Fock space

$$\hat{\mathcal{A}} = \exp(i\hat{\mathcal{Y}})$$

such that

$$\hat{\mathcal{A}}\mathbf{c}\hat{\mathcal{A}}^+ = \mathcal{A}^+\mathbf{c} = \boldsymbol{\gamma}$$

Proof:

$$\exp(D)E\exp(-D) = E + \sum_{n=1}^{\infty} \frac{1}{n!} [D, [D, \dots [D, E] \dots]]_n,$$

$$\hat{\mathcal{A}}\mathbf{c}\hat{\mathcal{A}}^+ = \mathbf{c} + \sum_{n=1}^{\infty} \frac{i^n}{n!} [\hat{\mathcal{Y}}, [\hat{\mathcal{Y}}, \dots [\hat{\mathcal{Y}}, \mathbf{c}] \dots]]_n.$$

$$\begin{aligned} [\hat{\mathcal{Y}}, c_m] &= \frac{1}{2} \sum_{kl} \mathcal{Y}_{kl} [c_k^+ c_l, c_m] = \frac{1}{2} \sum_{kl} \mathcal{Y}_{kl} (\mathcal{F}_{lm} c_k^+ - \delta_{km} c_l) \\ &= \frac{1}{2} \sum_k (\mathcal{F}\mathcal{Y}^T\mathcal{F} - \mathcal{Y})_{mk} c_k = - \sum_k \mathcal{Y}_{mk} c_k \end{aligned}$$

$$[\hat{\mathcal{Y}}, [\hat{\mathcal{Y}}, \dots [\hat{\mathcal{Y}}, c_m] \dots]]_n = \sum_k (-\mathcal{Y})_{mk}^n c_k$$

$$\hat{A}c_m\hat{A}^+ = \sum_k (\exp(-i\mathcal{Y}))_{mk} c_k = \sum_k \mathcal{A}_{mk}^+ c_k$$

The Bogoliubov transformation in the Fock space takes us from the old fermion operators to quasiparticles

$$\hat{A}a_{\mu_1}^+ \dots a_{\mu_A}^+ |0\rangle = \alpha_{\mu_1}^+ \dots \alpha_{\mu_A}^+ |\Phi\rangle.$$

$$|\Phi\rangle \equiv \hat{A}|0\rangle$$

Note that the Bogoliubov transformation is linear in a^+ and a . This guarantees that the Wick's theorem holds for all the vacua $|\Phi\rangle$ obtained from the real particle vacuum $|0\rangle$

Improper Bogoliubov Transformations

Consider the hermitian and unitary operator

$$\hat{A}^{(\mu)} \equiv a_{\mu}^{\dagger} + a_{\mu}$$

$$\hat{A}^{(\mu)} a_{\nu}^{\dagger} \hat{A}^{(\mu)\dagger} = \begin{cases} -a_{\nu}^{\dagger} & \text{for } \nu \neq \mu \\ a_{\nu} & \text{for } \nu = \mu. \end{cases}$$

We see that A is the Bogoliubov transformation in the Fock space. The corresponding matrix transformation can be written as:

$$A_{\nu\nu'}^{(\mu)} = \begin{cases} -1 & \text{for } \nu = \nu' \neq \mu \\ 0 & \text{other cases,} \end{cases}$$

$$B_{\nu\nu'}^{(\mu)} = \begin{cases} 1 & \text{for } \nu = \nu' = \mu \\ 0 & \text{other cases.} \end{cases}$$

$$\text{Det}(\mathcal{A}^{(\mu)}) = -1 \quad \Rightarrow \quad \text{improper Bogoliubov transformation}$$

All improper Bogoliubov transformations can be written as

$$\begin{aligned} \hat{A} &= \exp(i\hat{Y})\hat{A}^{(1)} \\ \mathcal{A} &= \exp(i\mathcal{Y})\mathcal{A}^{(1)} \end{aligned}$$

Let us introduce the particle-number-parity operator:

$$\hat{\pi}_N = \exp(i\pi\hat{N})$$

The operator $e^{i\hat{Y}}$ does not change parity π_N and $\hat{A}(\mu)$ changes it. Consequently.

$$\det \mathcal{A} = \pm 1 \quad \Leftrightarrow \quad \hat{\pi}_N \hat{A} = \pm \hat{A} \hat{\pi}_N$$

- Proper Bogoliubov transformations conserve the particle-number-parity
- Improper Bogoliubov transformations change the particle-number-parity

Product States -summary

- The simplest states in the Fock space that obey Wick's theorem. (It is easy to calculate expectation values of operators in product states!)
- Correspond to vacua of quasi-particle operators α_k
- Form a complete set in the Fock space
- They are eigenstates of one-body operators. (Describe non-interacting fermions)
- Can be used in perturbative calculations
- All product states can be written in a form

$$\left. \begin{array}{l} |\Phi\rangle_{\text{even}} \\ |\Phi\rangle_{\text{odd}} \end{array} \right\} = \exp(i\hat{Y}) \left\{ \begin{array}{l} 1 \\ a_1^+ \end{array} \right\} |0\rangle.$$

$$\begin{aligned} |\Phi\rangle_{\text{even}} &= \exp\left(\frac{i}{2} \sum_{kl} \mathcal{Y}_{kl} c_k^+ c_l\right) |0\rangle \\ &= \exp\left(i \sum_{\mu\nu} \left\{ \frac{1}{2} X_{\mu\nu}^+ a_\mu^+ a_\nu^+ + Y_{\mu\nu} a_\mu^+ a_\nu + \frac{1}{2} X_{\mu\nu} a_\mu a_\nu \right\} - \frac{i}{2} \text{Tr} Y\right) |0\rangle \end{aligned}$$

A parameterization of a product state... What is the number of parameters?

$$X^T = -X \Rightarrow M(M-1) \text{ real parameters}$$

$$Y^+ = Y \Rightarrow M^2 \text{ real parameters}$$

In reality, it is enough to use only $M(M-1)$ real parameters!

Ring and Schuck theorem

Every even-particle-number product state can be written as

$$|\Phi\rangle_{\text{even}} = e^{i\phi} \exp \left(\frac{i}{2} \sum_{\mu\nu} \{ T_{\mu\nu}^+ a_{\mu}^+ a_{\nu}^+ + T_{\mu\nu} a_{\mu} a_{\nu} \} \right) |0\rangle$$

where $T^T = -T$

(there is no need to introduce Y-matrices)

Thouless theorem (1960)

Every even product state, non-orthogonal to $|0\rangle$ can be uniquely written in the form:

$$|\Phi\rangle_{\text{even}} = \mathcal{N} \exp \left(-\frac{1}{2} \sum_{\mu\nu} Z_{\mu\nu}^+ a_{\mu}^+ a_{\nu}^+ \right) |0\rangle$$

$$Z^T = -Z$$

normalization
constant

Thouless
matrix

$$\cos(T^+ T)^{1/2} = (1 + Z^+ Z)^{-1/2}$$

Generalized coherent states

$$|Z\rangle = \exp\left(\frac{1}{2} \sum_{\mu\nu} Z_{\mu\nu}^* a_{\mu}^+ a_{\nu}^+\right) |0\rangle$$

A product state in the Thouless representation but not normalized

$$\langle Z|0\rangle = 1 \quad \text{a phase convention...}$$

Onishi Theorem

$$\langle Z'|Z\rangle = \det^{1/2}(1 + Z'Z^+)$$

Density Matrices

$$\rho_{\mu\nu} = \langle \Phi | a_\nu^\dagger a_\mu | \Phi \rangle = \overline{a_\nu^\dagger a_\mu}$$

$$\kappa_{\mu\nu} = \langle \Phi | a_\nu a_\mu | \Phi \rangle = \overline{a_\nu a_\mu}$$

$$\rho^\dagger = \rho,$$

$$\kappa^T = -\kappa$$

In principle, by measuring various one- and many-body observables, one can extract the whole density matrix.

$$\langle \Phi | \hat{F} | \Phi \rangle = \sum_{\mu\nu} F_{\mu\nu} \rho_{\nu\mu} = \text{Tr} F \rho$$

$$\langle \Phi | \hat{F} | \Phi \rangle = \sum_{\mu\mu'\nu\nu'} F_{\mu\mu'\nu\nu'} \left(\frac{1}{2} \rho_{\nu\mu} \rho_{\nu'\mu'} + \frac{1}{4} \kappa_{\mu'\mu}^+ \kappa_{\nu'\nu} \right)$$

In terms of the matrix Bogoliubov transformation:

$$\rho = B^* B^T,$$

$$\kappa = B^* A^T$$

The conditions

$$\rho^2 + \kappa \kappa^+ = \rho,$$

$$\rho \kappa - \kappa \rho^* = 0$$

is equivalent to the unitarity conditions between matrices A and B .