Coherent SU(2) states

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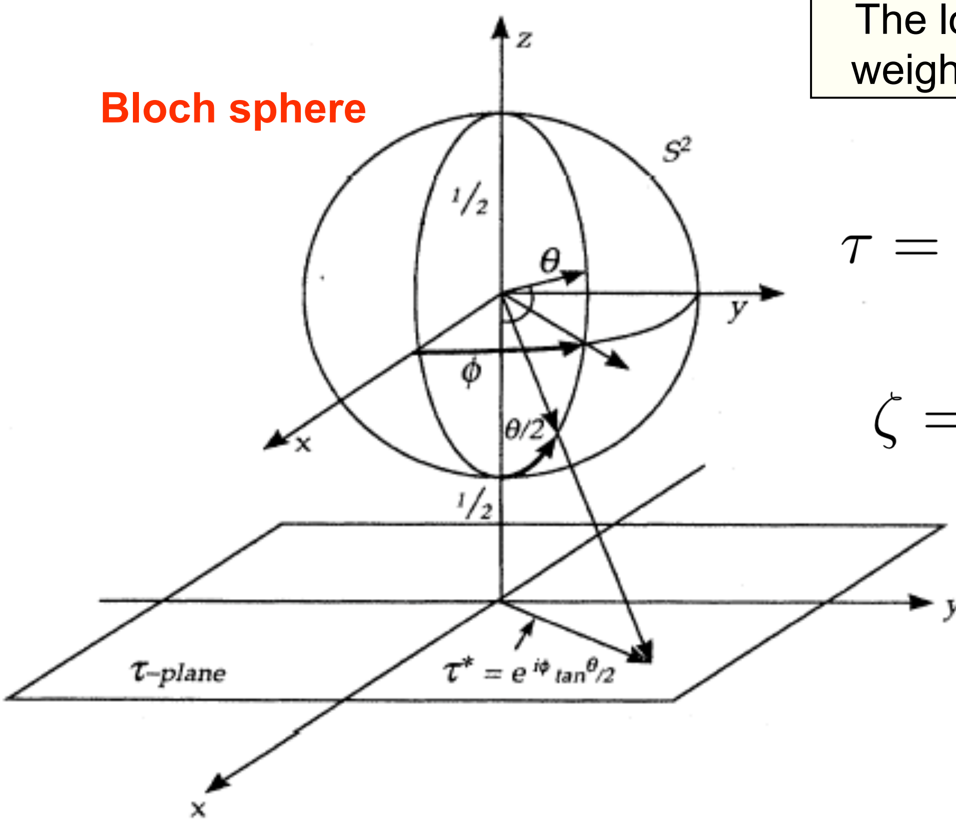
$$\hat{\mathbf{J}}^2|JM\rangle = J(J+1)|JM\rangle$$
$$\hat{J}_0|JM\rangle = M|JM\rangle$$

All the states can be obtained the raising J_- operator to the lowest-weight state J+M times:

$$|JM\rangle = \binom{2J}{J+M}^{-\frac{1}{2}} \frac{(\hat{J}_+)^{J+M}}{(J+M)!}|J-J\rangle$$

The lowest-weight state

Bloch sphere



$$\tau = \tan\frac{\theta}{2}e^{-i\phi}$$

$$\zeta = \frac{\theta}{2}e^{-i\phi}$$

FIG. 5. Geometry of SU(2) coherent states. The SU(2) coherent states map the two-dimensional sphere S^2 onto the complex plane. This is compact when one includes the point at infinity.

Any group of transformations g of SU(2) acting on the extremal (here: lowest-weight) state can be expressed as follows:

$$|J, \zeta\rangle = e^{\zeta \hat{J}_+ - \zeta^* \hat{J}_-} |J \; -J\rangle$$

Such states are called **spin (or atomic) coherent states**

The parameters ζ and ζ* are the geometrical conjugate variables of J_+ and J_- . The normalized coherent SU(2) states can be written as:

$$|J, \zeta\rangle = \frac{1}{(1+|\tau|^2)^J} e^{\tau \hat{J}_+} |J \; -J\rangle$$

Baker-Campbell-Hausdorff formulae

$$\begin{aligned} e^{\zeta \hat{J}_+ - \zeta^* \hat{J}_-} &= e^{\tau \hat{J}_+} e^{\ln(1+|\tau|^2)\hat{J}_0} e^{-\tau^* \hat{J}_-} \\ &= e^{-\tau^* \hat{J}_-} e^{-\ln(1+|\tau|^2)\hat{J}_0} e^{\tau \hat{J}_+} \end{aligned}$$

The atomic coherent states are in general not orthogonal, except for antipodal points:

$$|\langle J, \zeta' | J, \zeta\rangle|^2 = \left[\frac{1 + \boldsymbol{n}(\Omega') \cdot \boldsymbol{n}(\Omega)}{2}\right]^{2J} = \cos^{4J} \frac{\Theta}{2}$$

$\boldsymbol{n}(\Omega)$ - the unit vector representing the point on the Bloch sphere

Over-completeness

$$\frac{2J+1}{4\pi}\int d\Omega |J,\zeta\rangle\langle J,\zeta| = \sum_M |JM\rangle\langle JM| = 1$$

Generating functions

$$\langle\zeta|e^{\alpha_- \hat{J}_-} e^{\alpha_0 \hat{J}_0} e^{\alpha_+ \hat{J}_+}|\zeta\rangle =$$
$$= \frac{1}{(1+|\tau|^2)^{2J}} \left[e^{-\frac{1}{2}\alpha_0} + e^{\frac{1}{2}\alpha_0}(\tau^* + \alpha_-)(\tau + \alpha_+)\right]^{2J}$$

$$\langle\zeta|e^{\alpha_+ \hat{J}_+} e^{\alpha_0 \hat{J}_0} e^{\alpha_- \hat{J}_-}|\zeta\rangle =$$
$$= \frac{1}{(1+|\tau|^2)^{2J}} \left[e^{\frac{1}{2}\alpha_0}|\tau|^2 + e^{-\frac{1}{2}\alpha_0}(\alpha_+\tau^* + 1)(\alpha_-\tau + 1)\right]^{2J}$$

The matrix element of any complicated operator in the coherent states can be obtained by simple derivative computations of the generating functions.

Problems: (a) Calculate normalization of the coherent state using the method of generating functions
(b) Calculate the overlap between two coherent states