Fock Space: Second quantization representation

Let us define the **creation operators** acting on the vectors in the Fock space:

$$a^+_\mu \Phi_{\mu_1 \cdots \mu_A} \equiv \begin{cases} 0 & \text{if} \quad \mu \in \{\mu_i\} \\ \Phi_{\mu\mu_1 \cdots \mu_A} & \text{if} \quad \mu \notin \{\mu_i\} \end{cases}$$

a^+_μ takes us from $\mathcal{H}_A$ to $\mathcal{H}_{A+1}$

By calculating the scalar product of $a^+_\mu \Phi_{\nu_1 \cdots \nu_A}$

with an arbitrary state $\Phi_{\mu_1 \cdots \mu_{A+1}}$ it is easy to show that, for $a_\mu \equiv (a^+_\mu)^+$

$$a_\mu \Phi_{\mu_1 \cdots \mu_{A+1}} = \begin{cases} 0 & \text{if} \quad \mu \notin \{\mu_i\} \\ (-1)^{k+1} \Phi_{\mu_1 \cdots \mu_{k-1} \mu_{k+1} \cdots \mu_{A+1}} & \text{if} \quad \mu = \mu_k \end{cases}$$

μ is absent!

a_μ takes us from $\mathcal{H}_{A+1}$ to $\mathcal{H}_A$

… thus is called an "**annihilation operator**" let us consider a_μ^+ and a_ν^+ $(\mu \neq \nu)$

and $\Phi_{\mu_1 \cdots \mu_A}$ $(\mu \neq \mu_i, \nu \neq \mu_i)$

$$
\begin{aligned}
a_\mu^+ a_\nu^+ \Phi_{\mu_1 \cdots \mu_A} &= \Phi_{\mu\nu\mu_1 \cdots \mu_A} = \\
&= -\Phi_{\nu\mu\mu_1 \cdots \mu_A} = -a_\nu^+ a_\mu^+ \Phi_{\mu_1 \cdots \mu_A}
\end{aligned}
$$

$$
\Longrightarrow \quad a_\mu^+ a_\nu^+ = -a_\nu^+ a_\mu^+
$$

$$
a_\nu a_\mu = -a_\mu a_\nu
$$

Similarly, for $\nu=\mu_k$ and $\nu \neq \mu_i$, one obtains

$$
a_\mu^+ a_\nu \Phi_{\mu_1 \cdots \mu_A} = (-1)^{k+1} \Phi_{\mu\mu_1 \cdots \mu_{k-1}\mu_{k+1} \cdots \mu_A} =
$$

$$
-a_\nu \Phi_{\mu\mu_1 \cdots \mu_A} = -a_\nu a_\mu^+ \Phi_{\mu_1 \cdots \mu_A}
$$

For $\nu=\mu=\mu_k$:

$$
\begin{aligned}
a_\nu^+ a_\nu \Phi_{\mu_1 \cdots \mu_A} &= (-1)^{k+1} \Phi_{\nu\mu_1 \cdots \mu_{k-1}\mu_{k+1} \cdots \mu_A} = \\
&= \Phi_{\mu_1 \cdots \mu_A} = \Phi_{\mu_1 \cdots \mu_A} - a_\nu a_\nu^+ \Phi_{\mu_1 \cdots \mu_A}
\end{aligned}
$$

$$
\Longrightarrow \quad a_\mu^+ a_\nu = \delta_{\mu\nu} - a_\nu a_\mu^+
$$

$$\boxed{\begin{aligned}\{a_\mu^+, a_\nu^+\} &= 0\\ \{a_\mu, a_\nu\} &= 0\\ \{a_\mu^+, a_\nu\} &= \delta_{\mu\nu}\end{aligned}} \qquad (1)$$

Consequently, $(a_\mu^+)^2 = 0\,,\ (a_\mu)^2 = 0$

(The Pauli Principle!)

In addition, we have to consider a case of the vacuum vector:

$$a_\mu^+ \Phi_{vac} = \Phi_\mu = \phi_\mu(x)$$

$$a_\mu \Phi_{vac} = 0$$

The Fock space defines a^+ and a !!!

But…

…the Fock space can be defined through a^+ and a !!!

Namely, given a set of *M* operators a_μ^+ and a_μ that meet the anticommutation rules (1) in a given Hilbert space, and assuming that there exists a vacuum state $|0\rangle$, such that

$$a_\mu|0\rangle = 0 \qquad \text{for any } \mu$$

… one can show that the set of states

$$|\mu_1 \cdots \mu_a\rangle \equiv a^+_{\mu_1} \cdots a^+_{\mu_A}|0\rangle, \ \ \mu_1 < \cdots < \mu_A$$

where *A=0,1,…,M,* is a basis of a space that is isomorphic with the Fock space, that is:

$$\Phi_{\mu_1 \cdots \mu_A} \leftrightarrow |\mu_1 \cdots \mu_A\rangle$$
$$\Phi_{vac} \leftrightarrow |0\rangle$$

The vectors $|\mu_1 \cdots \mu_A\rangle$ form a basis in the Fock Space (or, more precisely, a basis of the representation of the Fock Space), which is called the **second quantization representation.**

Properties of the vacuum

1. For a given set of creation and annihilation operators acting in a Fock space, there exists only one vacuum.

Proof: let us assume that in addition to $|0\rangle, \quad \langle 0|0\rangle = 1,$

there exists another vacuum $|0'\rangle, \quad \langle 0'|0'\rangle = 1,$

such that $a_\mu |0'\rangle = 0$ for any μ. Of course,

$$\langle 0'|a^+_{\mu_1} a^+_{\mu_2} \cdots a^+_{\mu_A}|0\rangle = 0 \;\; (A = 1, 2, \cdots M)$$

If $\langle 0'|0\rangle = 0$ then the vector $|0'\rangle$ is orthogonal to <u>all the vectors</u> in the Fock space; hence, it has to be a zero vector. Consequently, $\langle 0'|0\rangle \neq 0.$

Let us, therefore consider the vector $|0'\rangle - \langle 0'|0\rangle|0\rangle$ that is orthogonal to $|0\rangle$ and is also a vacuum. Hence, it has to be a zero vector. Thus, finally,

$$|0'\rangle = \underbrace{\boxed{\langle 0'|0\rangle}}_{\text{a phase…}}|0\rangle$$