

There exist many representations of the Fock space !

Homework: Assuming that $\langle 0|0\rangle = 1$ show that the state vectors in the second quantization representation

$a_{\mu_1}^+ a_{\mu_2}^+ \cdots a_{\mu_A}^+ |0\rangle$ are orthonormal.

2. Existence of vacuum

If $a_{\mu_1} a_{\mu_2} \cdots a_{\mu_M} \neq 0$ then there exists a vacuum state.

$$|0\rangle \equiv \mathcal{N} a_{\mu_1} a_{\mu_2} \cdots a_{\mu_M} |\Psi\rangle$$

$$a_{\mu} |0\rangle = 0 \quad \text{for any } \mu.$$

3. Unitary transformations in the Fock space

Let us assume that in a given Fock space there exist two sets of creation/annihilation operators together with the corresponding vacua:

$$\{a_{\mu}^+, a_{\nu}^+\} = \{a_{\mu}, a_{\nu}\} = 0, \quad \{a_{\mu}, a_{\nu}^+\} = \delta_{\mu\nu}, \quad a_{\mu} |0_{\alpha}\rangle = 0$$

$$\{\alpha_{\mu}^+, \alpha_{\nu}^+\} = \{\alpha_{\mu}, \alpha_{\nu}\} = 0, \quad \{\alpha_{\mu}, \alpha_{\nu}^+\} = \delta_{\mu\nu}, \quad \alpha_{\mu} |0_{\alpha}\rangle = 0$$

There exists one and only one unitary transformation between these two sets.

Let us introduce two orthogonal bases:

$$a_{\mu_1}^+ \dots a_{\mu_A}^+ |0_a\rangle \quad \text{and} \quad \alpha_{\mu_1}^+ \dots \alpha_{\mu_A}^+ |0_\alpha\rangle$$

The transformation operator that takes us from one basis to another must be unitary:

$$\hat{U} a_{\mu_1}^+ \dots a_{\mu_A}^+ |0_a\rangle = \alpha_{\mu_1}^+ \dots \alpha_{\mu_A}^+ |0_\alpha\rangle$$

Or, equivalently:

$$\left(\hat{U} a_{\mu_1}^+ \hat{U}^+ \right) \dots \left(\hat{U} a_{\mu_A}^+ \hat{U}^+ \right) \hat{U} |0_a\rangle = \alpha_{\mu_1}^+ \dots \alpha_{\mu_A}^+ |0_\alpha\rangle$$

As this must hold for any set of indices, one gets:

$$\hat{U} a_{\mu}^+ \hat{U}^+ = \alpha_{\mu}^+ \quad \text{and} \quad \hat{U} |0_a\rangle = |0_\alpha\rangle$$

One can thus conclude that all sets of creation/annihilation operators can be obtained through unitary transformations of one set a_{μ}^+ and their vacua are unitary transformations of $|0\rangle$

Is this transformation univocal? Assume that there are two such transformations \hat{U}_1 and \hat{U}_2

$$\hat{U}_1 a_{\mu}^+ \hat{U}_1^+ = \alpha_{\mu}^+ \quad , \quad \hat{U}_1 |0_a\rangle = |0_\alpha\rangle$$

$$\hat{U}_2 a_{\mu}^+ \hat{U}_2^+ = \alpha_{\mu}^+ \quad , \quad \hat{U}_2 |0_a\rangle = |0_\alpha\rangle$$

Let us now introduce the operator $\hat{U}_3 \equiv \hat{U}_2^+ \hat{U}_1$

$$\hat{U}_3 a_{\mu}^+ \hat{U}_3^+ = a_{\mu}^+ \quad , \quad \hat{U}_3 |0_a\rangle = |0_a\rangle$$

Therefore $\hat{U}_3 = 1$ i.e, $\hat{U}_1 = \hat{U}_2$

Another important observation is that any non-zero vector $|\Psi\rangle$ in the Fock space represents a vacuum for some set of creation/annihilation operators.

$$\hat{U}|0\rangle = |\Psi\rangle$$

Example: quasiparticle vacuum

Second quantization representation for the operators in the Fock space

Let us first define the particle number operator: $\hat{N}_\nu \equiv a_\nu^\dagger a_\nu$

$$\hat{N}_\nu |\mu_1 \dots \mu_A\rangle = n_\nu |\mu_1 \dots \mu_A\rangle,$$

n_ν is equal to 1 if $\nu \in \{\mu_i\}$ and 0 otherwise. It is called occupation number. These occupations define the so-called occupation number representation of the Fock space:

$$|n_1 \dots n_M\rangle \equiv (a_1^\dagger)^{n_1} \dots (a_M^\dagger)^{n_M} |0\rangle$$

The total particle number (or fermion number) operator is:

$$\hat{N} \equiv \sum_{\nu=1}^M \hat{N}_\nu = \sum_{\nu=1}^M a_\nu^\dagger a_\nu$$

$$\hat{N} |\mu_1 \dots \mu_A\rangle = A |\mu_1 \dots \mu_A\rangle$$

Another useful operator:

$$\hat{P}_{|0\rangle} = \prod_{\nu=1}^M (1 - \hat{N}_\nu) = \prod_{\nu=1}^M (1 - a_\nu^\dagger a_\nu) \quad \hat{P}_{|0\rangle} = |0\rangle\langle 0|$$

It projects out the vacuum configuration

What is the form of operators in the Fock space?

$$\hat{F} = \sum_{mn=1}^{2^M} |m\rangle F_{mn} \langle n|$$

matrix element of operator
in the Fock space

In the second quantization representation, an operator acting in the Fock space can be written as a sum of products of creation and annihilation operators!

Let us consider a k -particle operator acting in an A -particle subspace of the Fock space ($A \geq k$). In the coordinate representation, this operator can be written as:

$$\hat{F} = \sum_{j_1 < \dots < j_K}^A f(x_{j_1}, \dots, x_{j_K})$$

a function or integro-differential operator

Let us now calculate the matrix element:

$$\begin{aligned} & \int dx_1 \dots dx_A \Phi_{\mu_1 \dots \mu_A}^*(x_1, \dots, x_A) \hat{F} \Phi_{\nu_1 \dots \nu_A}(x_1, \dots, x_A) \\ &= A!^{-1} \int dx_1 \dots dx_A \sum (-1)^P \phi_{\mu_{i_1}}^*(x_1) \dots \phi_{\mu_{i_A}}^*(x_A) \\ & \quad \times \sum_{j_1 < \dots < j_K}^M f(x_{j_1}, \dots, x_{j_K}) \sum_{P'} (-1)^{P'} \phi_{\nu_{i'_1}}^*(x_1) \dots \phi_{\nu_{i'_A}}^*(x_A). \end{aligned}$$

In the above expression, we can immediately integrate over $A-k$ variables

$$F_{\mu_1 \dots \mu_K \nu_1 \dots \nu_K} \equiv \int dx_1 \dots dx_K \Phi_{\mu_1 \dots \mu_K}^* \hat{F} \Phi_{\nu_1 \dots \nu_K}$$

Antisymmetrized matrix element

$$\begin{aligned} F_{\mu_1 \dots \mu_K \nu_1 \dots \nu_K} &= \int dx_1 \dots dx_K \phi_{\mu_1}^*(x_1) \dots \phi_{\mu_K}^*(x_K) \\ & \quad \times f(x_1, \dots, x_K) \sum_P (-1)^P \phi_{\nu_{i_1}}(x_1) \dots \phi_{\nu_{i_K}}(x_K) \end{aligned}$$

The summation over all permutations gave $A!$ identical terms that cancelled out the normalization factor!

Examples

a) One-body matrix element

$$F_{\mu\nu} = \int dx \phi_{\mu}^*(x) f(x) \phi_{\nu}(x)$$

b) Two-body matrix element

$$F_{\mu\mu'\nu\nu'} = \int dx dx' \phi_{\mu}^*(x) \phi_{\mu'}^*(x') f(x, x') \\ \times (\phi_{\nu}(x) \phi_{\nu'}(x') - \phi_{\nu'}(x) \phi_{\nu}(x'))$$

In the second quantization representation, a k -body operator is completely determined through its antisymmetrized matrix elements:

$$\hat{F} = (K!)^{-2} \sum_{\mu_1 \dots \mu_K \nu_1 \dots \nu_K} F_{\mu_1 \dots \mu_K \nu_1 \dots \nu_K} a_{\mu_1}^+ \dots a_{\mu_K}^+ a_{\nu_K} \dots a_{\nu_1}$$

Examples

a) One-body operator

$$\hat{F} = \sum_{\mu\nu} F_{\mu\nu} a_{\mu}^+ a_{\nu},$$

b) Two-body operator

$$\hat{F} = \frac{1}{4} \sum_{\mu\mu'\nu\nu'} F_{\mu\mu'\nu\nu'} a_{\mu}^+ a_{\mu'}^+ a_{\nu'} a_{\nu}$$