Seniority Model

The seniority model provides a simple model for pairing phenomena.
System: N fermions in a single j-shell.
Hamiltonian:

$$\begin{aligned} H &= -G \sum_{m,m'>0} \hat{a}^\dagger_m \hat{a}^\dagger_{-m} \hat{a}_{-m'} \hat{a}_{m'} \\ &= -G \hat{S}_+ \hat{S}_- \end{aligned}$$

where

$$\hat{S}_+ = \sum_{m>0} \hat{a}^\dagger_m \hat{a}^\dagger_{-m} \quad \text{and} \quad \hat{S}_- = \left(\hat{S}_+\right)^\dagger .$$

Introduce *quasi-spin operators*

$$\begin{aligned} \hat{s}^{(m)}_+ &= \hat{a}^\dagger_m \hat{a}^\dagger_{-m} \\ \hat{s}^{(m)}_- &= \hat{a}_{-m} \hat{a}_m \\ \hat{s}^{(m)}_0 &= \frac{1}{2}\left(\hat{a}^\dagger_m \hat{a}_m + \hat{a}^\dagger_{-m} \hat{a}_{-m} - 1\right) \end{aligned}$$

and find angular momentum commutation relations!

$$\begin{aligned} \left[\hat{s}^{(m)}_+, \hat{s}^{(m)}_-\right] &= 2\hat{s}^{(m)}_0 \\ \left[\hat{s}^{(m)}_0, \hat{s}^{(m)}_+\right] &= \hat{s}^{(m)}_+ \\ \left[\hat{s}^{(m)}_0, \hat{s}^{(m)}_-\right] &= -\hat{s}^{(m)}_- \end{aligned}$$

Rewrite Hamiltonian as

$$H = -G\left(\vec{S}\cdot\vec{S} - \hat{S}_0^2 + \hat{S}_0\right)$$

in terms of total quasi-spin

$$\vec{S} = \sum_{m>0} \vec{s}^{(m)}.$$

and total z-component of quasi spin

$$\hat{S}_0 = \sum_{m>0} \hat{s}_0^{(m)} = \frac{1}{2}(\hat{N} - \Omega).$$

Here, $\Omega = j + 1/2$ is the maximal number of pairs for a single j-shell. The eigenvalues S of total quasi-spin are

$$S = \frac{1}{2}|N - \Omega|, \dots, \frac{1}{2}\Omega - 1, \frac{1}{2}\Omega.$$

Thus, the energies of the seniority model are

$$E(S, N) = -G\left[S(S+1) - \frac{1}{4}(N - \Omega)^2 + \frac{1}{2}(N - \Omega)\right].$$

Alternatively, one uses the *seniority* quantum number $s = \Omega - 2S$

$$E(s, N) = -\frac{G}{4}\left[s^2 - 2s(\Omega + 1) + 2N(\Omega + 1) - N^2\right].$$

Note:

- s counts number of unpaired nucleons.
- ground state has minimal seniority $s = 0$ (or maximal quasi spin $S = \Omega/2$)
- for fixed N, excitations depend only on seniority quantum number
- $E(N, s = 2) - E(N, s = 0) = G\Omega$

Two-particle spectrum of pure pairing force: $J = 0$ ground state is separated from degenerate $J = 2, 4, 6, \dots, 2j - 1$ levels.

Pair vibrational spectrum: $E(N, s = N) - E(N, s = 0) \approx G\Omega N/2$ for $N \ll \Omega$. (Binding increases linearly with number of pairs.)